Disagreement Under Almost Common Knowledge of Rationality

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Paradox

Aumann (1976)

If two individuals share the same prior and have common knowledge among them of

- their information partitions
- their rationality (that is, both agents use Bayes rule to update their beliefs)
- their posteriors

then these posteriors must be the same (they cannot agree to disagree)

Question : Why do we disagree?

Our answer : People disagree because they doubt each other's rationality (and they doubt each other's rationality because they disagree)

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Robustness:

- Geanakoplos and Polemarchakis (1982): c.k. of rationality + communication of posteriors
- Cave (1983), Bacharach (1985): c.k. of rationality + communication of decisions among like-minded agents

Result (informally)

People disagree because they stop “hearing” each other, stop updating their beliefs. That happens if we relax the assumption of **common knowledge** of rationality.

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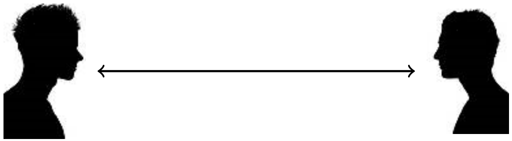

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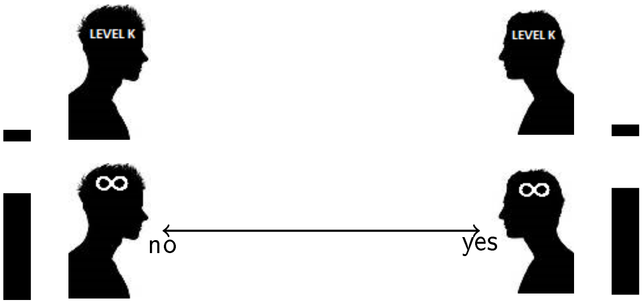

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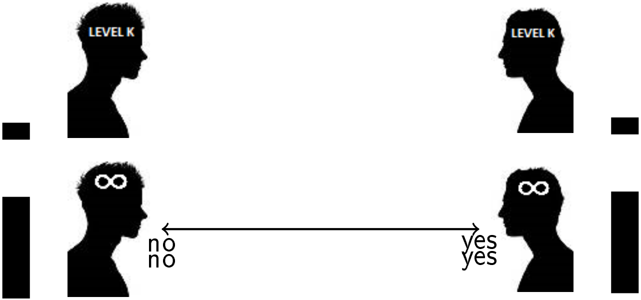

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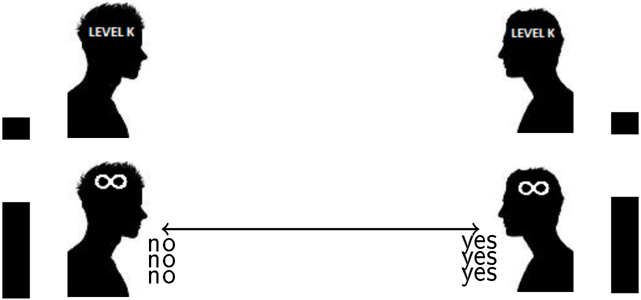

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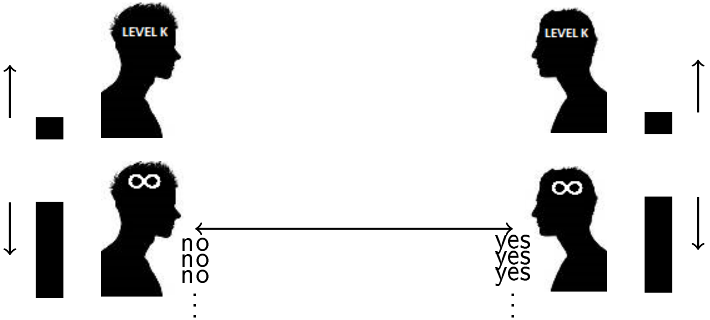

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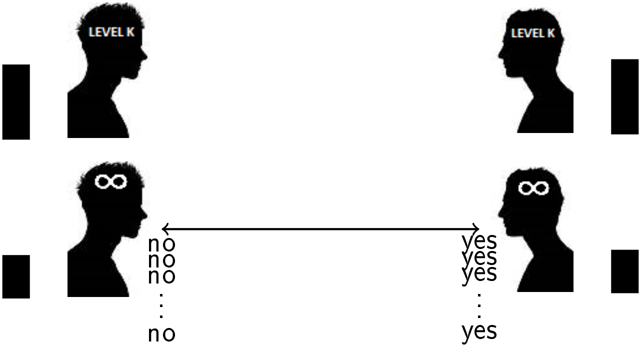

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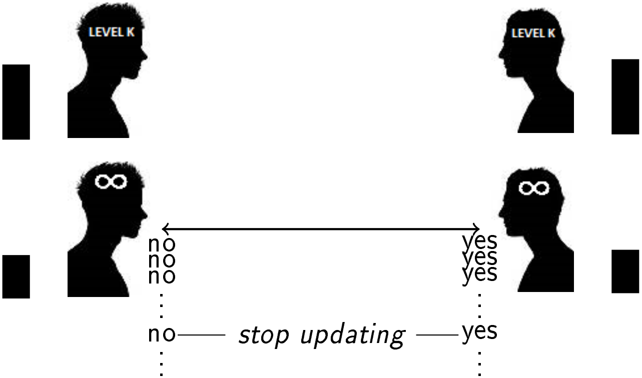

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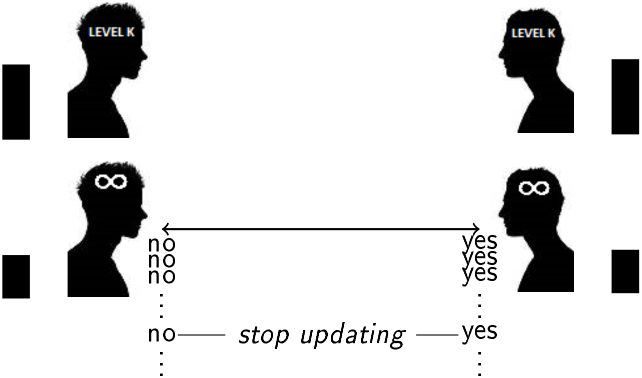

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Model

Communication game

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- Two agents exchange their guesses (0 or 1) about the state each period $t = 1, 2, \ldots$
  - assume the agents guess the state sincerely
- Period 1: common prior + private signals

Model

Belief formation

Based on the cognitive hierarchical (CH) theory proposed in Camerer et al (2004)

Level 0 agent randomizes each period: $\begin{cases} 0, & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$

Level 1 agent thinks that his opponent is of level 0

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Parameter of the model: function f — probability distribution over the levels of thinking

$$f(k) \in [0,1], \quad \sum_{k=0}^{+\infty} f(k) = 1$$

Level $k \geq 2$ agent thinks that the level of thinking of his opponent is distributed over 0 through $k-1$ according to the distribution f truncated at level $k-1$.

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Novelty: f might be inconsistent with realized distribution. We assume both agents are of level ∞. Alternative interpretation: we make predictions only for ∞-level agents.

Result (formally)

Theorem

For any level $k \geq 1$, any positive amount of doubt $\varepsilon \in (0, 1)$, any "correct" posterior $\rho = \mathbb{P}\left[\theta = 1 \mid s^{(1)}, s^{(2)}\right] \in (0, 0.5)$, we can find a prior, a signal structure $F_i^{(\theta)}$, $i \in \{1, 2\}$, $\theta \in \{0, 1\}$, and the realization of the signals $s^{(1)}$ and $s^{(2)}$ such that the ∞-level agents with $f(0) = \ldots = f(k-1) = 0$ and $f(k) = \varepsilon$ permanently disagree with each other.

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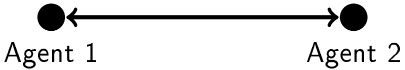
"correct" posterior belief $\rho \in (0, 0.5)$: $0.5 - \rho =$ ex post disagreement

$\Downarrow$

ex ante disagreement needed to guarantee

this ex post disagreemen

Example 1



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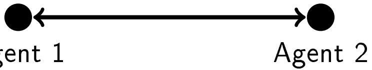

Signals: 0 1

Common prior: ρ. Signals are binary and iid:

$$\mathbb{P}[s = \theta \mid \theta] = q \in (0.5, 1), \quad \mathbb{P}[s = 1 - \theta \mid \theta] = 1 - q$$

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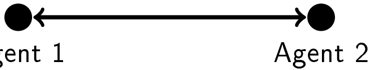

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1

Assume $\rho + q > 1 \Rightarrow$ always optimal to guess the signal in period 1: $\mathbb{P}\left[\theta = s \mid s\right] > 0.5$

Example 1

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	
	⋮	
	0	

Prior is biased towards 0 $\Rightarrow$ agent 1 always reports 0

Example 1

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	?
	⋮	
	0	

Agent 2:

$f(0)$ → agent 1 is of level 0 $\Rightarrow$ best guess is 1

$1 - f(0)$ → agent 1 is of level ≥ 1 $\Rightarrow$ best guess is 0

Example 1

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	? $\leftarrow$ depends on $f(0)$
	$\vdots$	
	0	

Agent 2:

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$1 - f(0)$ $\rightarrow$ agent 1 is of level $\geq 1 \Rightarrow$ best guess is 0

Example 1

	Agent 1	Agent 2	
Signals:	0	1	
Period 1:	0	1	
Period 2:	0	?	$\leftarrow$ depends on $f(0)$
	$\vdots$	$\vdots$	
	0	0	$\leftarrow$ agreement is inevitable

$$\mathbb{P}\left[s^{(2)} = 1,\ y_1^{(1)} = \ldots = y_{t-1}^{(1)} = 0 \mid \text{agent 1 is of level 0}\right] \underset{t\to\infty}{\longrightarrow} 0$$

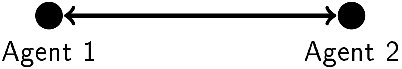
$\Rightarrow$ eventually, agent 2 learns that agent 1 is of level ≥ 1

Example 1

	Agent 1	Agent 2	
Signals:	0	1	
Period 1:	0	1	
Period 2:	0	?	$\leftarrow$ depends on $f(0)$
	$\vdots$	$\vdots$	
	0	0	$\leftarrow$ agreement is inevitable

1. Absence of c.k. of rationality does not necessarily lead to disagreement
2. Agents might learn each other's level of rationality

Example 2



Signals: $\{0, 0\}$ $\{1, 1\}$

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$$\mathbb{P}\left[s = \theta \mid \theta\right] = q \in (0.5, 1), \quad \mathbb{P}\left[s = 1 - \theta \mid \theta\right] = 1 - q$$

$\Rightarrow$ each agent receives $s = s_1 + s_2 \sim B(2, q)$

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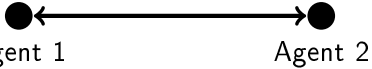

	Agent 1	Agent 2	
Signals:	$\{0, 0\}$	$\{1, 1\}$	
Period 1:	0	1	$\longleftarrow \rho + q > 1$

Example 2

	Agent 1	Agent 2	
Signals:	$\{0, 0\}$	$\{1, 1\}$	
Period 1:	0	1	$\longleftarrow \rho + q > 1$
Period 2:	0		
Period 3:	0		
	$\vdots$		
	0		

Prior is biased towards 0 $\Rightarrow$ agent 1 always reports 0

Example 2

	Agent 1	Agent 2	
Signals:	$\{0, 0\}$	$\{1, 1\}$	
Period 1:	0	1	$\longleftarrow \rho + q > 1$
Period 2:	0		
Period 3:	0		
	$\vdots$		
	0		

Assume $f(0) = 0$

Example 1 $\Rightarrow$ f(0) only impends learning, though not causes disagreement per se.
Example 2 shows disagreement $\Rightarrow$ setting $f(0) = 0$ makes our argument stronger

Example 2

	Agent 1	Agent 2	
Signals:	$\{0, 0\}$	$\{1, 1\}$	
Period 1:	0	1	$\longleftarrow \rho + q > 1$
Period 2:	0	1	
Period 3:	0		
	$\vdots$		
	0		

Assume $f(0) = 0$

Agent 2: agent 1 is of level $\geq 1 \Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1$

$$\mathbb{P}\left[\theta = 1 \mid s_1^{(2)} = s_2^{(2)} = 1,\ s_1^{(1)} + s_2^{(1)} \leq 1\right] > \frac{1}{2}$$

Example 2

	Agent 1	Agent 2	
Signals:	$\{0,0\}$	$\{1,1\}$	
Period 1:	0	1	$\longleftarrow \rho + q > 1$
Period 2:	0	1	
Period 3:	0	1	$\longleftarrow$ if $f(1) > \frac{1}{2\left(\frac{\rho+q-1}{(1-2\rho)(1-q)q}+1\right)}$
	$\vdots$		
	0		

Assume $f(0) = 0$

$f(1)$ → agent 1 is of level 1 $\Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1 \Rightarrow$ best guess is 1

$1 - f(1)$ → agent 1 is of level $\geq 2 \Rightarrow s_1^{(1)} = s_2^{(1)} = 0 \Rightarrow$ best guess is 0

Example 2

	Agent 1	Agent 2	
Signals:	$\{0,0\}$	$\{1,1\}$	
Period 1:	0	1	$\longleftarrow \rho + q > 1$
Period 2:	0	1	
Period 3:	0	1	permanent disagreement if $f(1) > \frac{1}{2\left(\frac{\rho+q-1}{(1-2\rho)(1-q)q}+1\right)}$
	$\vdots$	$\vdots$	
	0	1	

Assume $f(0) = 0$

Agent 2 knows:

- level 1 agent 1 stops updating after period 1
- level ≥ 2 agent 1 stops updating after period 2
- $\Rightarrow$ agent 1 stops updating after period 2

$\Rightarrow$ agent 2 stops updating after period 3

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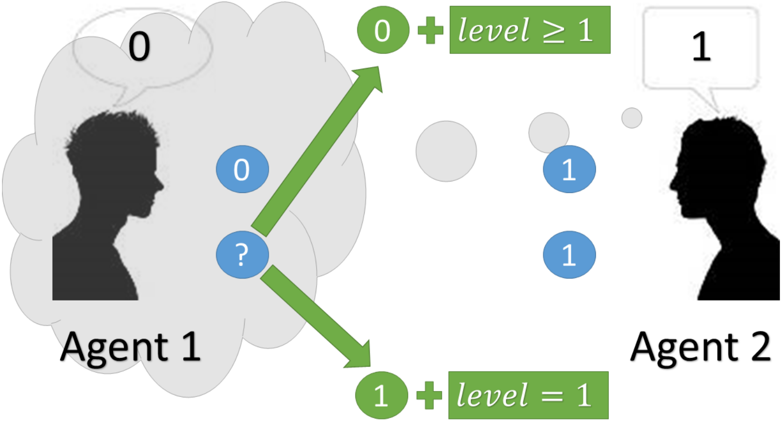

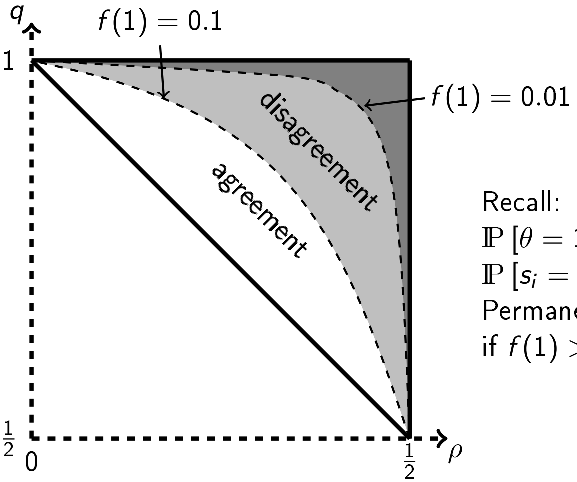

Recall:

$\mathbb{P}[\theta = 1] = \rho$

$\mathbb{P}[s_i = \theta \mid \theta] = q$

Permanent disagreement

if $f(1) > \frac{1}{2\left(\frac{\rho+q-1}{(1-2\rho)(1-q)q}+1\right)}$

The size of the doubt $f(1)$ can be arbitrary small and the correct posterior ρ can be arbitrary close to 0: for any $f(1) \in (0,1)$ and any $\rho \in (0, 0.5)$ we can find $q \in (0.5, 1)$ (characterizes ex ante disagreement) such that the agents permanently disagree with each other for some signals realizations.

Conclusion

Paradox

Aumann (1976): impossibility to agree to disagree

Our result

Model with almost common knowledge of rationality where disagreement is possible