

Cognitive Hierarchical Model in Networks

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4th Lancaster Game Theory Conference
2 November 2018

Paradox

- ▶ Aumann (1976): it is impossible to agree to disagree \Rightarrow No trade theorem
- ▶ Real world proves otherwise: we disagree and we trade

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Idea

Deviate from Aumann just a little bit \Rightarrow disagreement is possible

Very preliminary, your comments are very welcome!

Agreeing to Disagree

Aumann (1976)

If two individuals share the same prior and have common knowledge among them of

- ▶ their information partitions
- ▶ their rationality (that is, both agents use Bayes rule to update their beliefs)
- ▶ their posteriors

then these posteriors must be the same (they cannot agree to disagree)

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Cave (1983), Bacharach (1985): communication of decisions among like-minded agents

Gale and Kariv (2003): communication of decisions in a sufficiently connected network

Existing literature: Agreement result is robust to relaxing

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Ellison and Fudenberg (1993), DeMarzo, Vayanos, and Zwiebel (2003), Golub and Jackson (2010), Molavi, Tahbaz-Salehi and Jadbabaie (2017) etc : agreement for **boundedly rational** agents

Existing literature: Agreement result is robust to relaxing

Aumann (1976): c.k. of [✓]rationality + c.k. of [✓]posteriors [✓]

This paper: but not to *common knowledge* of rationality.

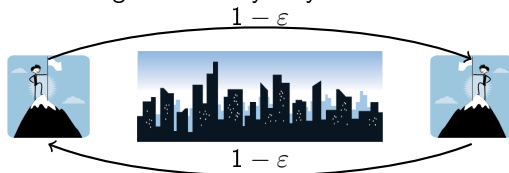
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Intuition: in the spirit of **Rubinstein (1989) Email Game** who shows that “the game-theoretic prediction for the almost common knowledge situation is very different from the situation with common knowledge”

Coordination game: victory only if both armies attack



Eventually, the message is lost with probability $1 - (1 - \epsilon)^t \xrightarrow{t \rightarrow \infty} 1$

Once the message is not received, it is not optimal to attack:

$$\underbrace{\epsilon}_{\text{initial message is lost}} \Rightarrow \text{other army does not attack} > \underbrace{(1 - \epsilon)\epsilon}_{\text{returned message is lost}}$$

Outline

Model

Example 1

Example 2

Example 3

Example 4

Model

State and signals

- ▶ state: $\theta \in \{0, 1\}$
- ▶ common prior: $\mathbb{P}[\theta = 1] = p \in (0, 0.5)$

Model

State and signals

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- ▶ common prior: $\mathbb{P}[\theta = 1] = p \in (0, 0.5)$
- ▶ private signals: independent conditional on θ

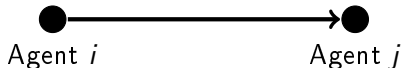
$$s \in \{0, 1\} \quad \mathbb{P}[s = \theta \mid \theta] = q \in (0.5, 1)$$

- ▶ assume $\mathbb{P}[\theta = s \mid s] > 0.5$, or equivalently $p + q > 1$

Model

Communication game

- ▶ directed network:



means *Agent j listens to Agent i*

- ▶ at each period $t \geq 1$, each agent guesses θ sincerely
 - ▶ assumption: no strategic considerations
- ▶ if agent j listens to agent i , then agent j observes all guesses of agent i from all **past** periods

Model

Belief formation

Based on the **cognitive hierarchical (CH) theory** proposed in Camerer et al (2004)

Level 0 agent randomizes each period: $\begin{cases} 0, & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$

Level 1 agent thinks that all other players are of level 0

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Parameter of the model: function f — probability distribution over the levels of thinking

$$f(k) \in [0, 1], \quad \sum_{k=0}^{+\infty} f(k) = 1$$

Level $k \geq 2$ agent thinks that the levels of thinking of all other players are independently distributed over 0 through $k - 1$ according to the distribution f truncated at level $k - 1$.

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Novelty: f might be inconsistent with realized distribution. We assume **everybody is of level ∞**

Model

Discussion

- ▶ Two levels of uncertainty: **state of the world** and **realized levels of thinking**.
- ▶ Agents learn about **both** over time

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- ▶ **Take** the distribution of signals such that agents eventually **agree** on the true state when it is c.k. that they are rational, that is **when $f(+\infty) = 1$**

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- ▶ Suppose **$f(+\infty) = 1 - \varepsilon$** for $\varepsilon > 0$ very small

Model

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- ▶ Agents learn about **both** over time
- ▶ **Take** the distribution of signals such that agents eventually **agree** on the true state when it is c.k. that they are rational, that is **when** $f(+\infty) = 1$
- ▶ Suppose $f(+\infty) = 1 - \varepsilon$ for $\varepsilon > 0$ very small
- ▶ *Is it possible that agents not only fail to learn that everybody is of level ∞ , but also **permanently disagree** on the state?*

Outline

Model

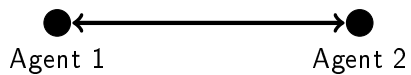
Example 1

Example 2

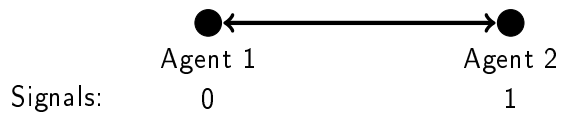
Example 3

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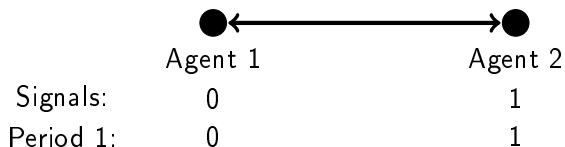
Example 1



Example 1

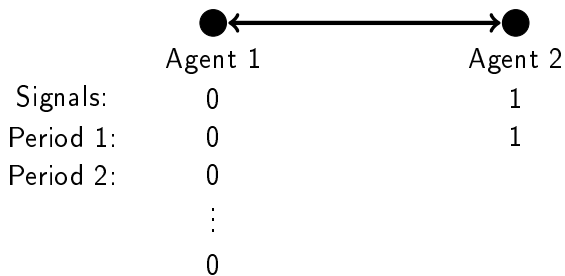


Example 1




Recall: $p + q > 1 \Rightarrow$ always optimal to guess the signal in period 1:
 $\mathbb{P}[\theta = s \mid s] > 0.5$

Example 1



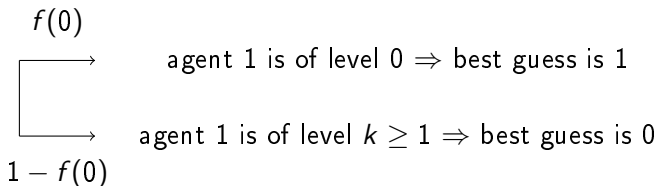
Recall: prior is biased towards 0: $\mathbb{P}[\theta = 1] = p < 0.5 \Rightarrow$ agent 1 always reports 0

Example 1

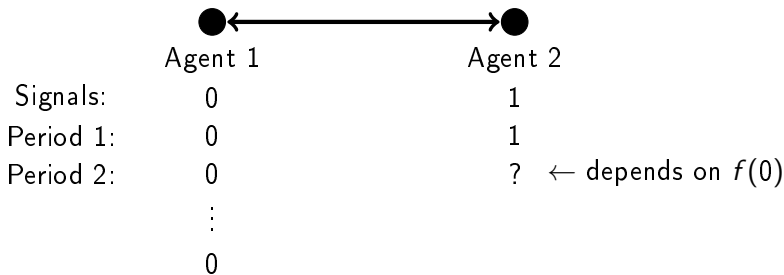


	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	?
	⋮	
	0	

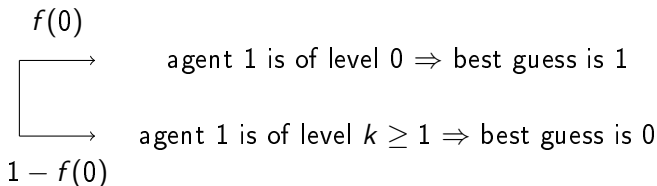
Agent 2:




Example 1



Agent 2:




Example 1



	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	? ← depends on $f(0)$
	⋮	⋮
	0	0 ← agreement is inevitable

$\mathbb{P} \left[s^{(2)} = 1, y_1^{(1)} = \dots = y_{t-1}^{(1)} = 0 \mid \text{agent 1 is of level } 0 \right] \xrightarrow[t \rightarrow \infty]{} 0$
 \Rightarrow eventually, agent 2 **learns** that agent 1 is of level $k \geq 1$

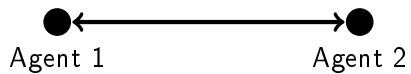
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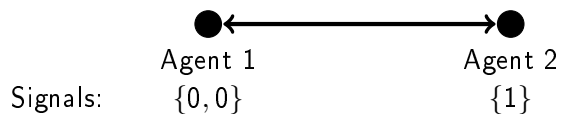
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	0	0 ← agreement is inevitable

1. Absence of c.k. of rationality does **not** necessarily leads to disagreement
2. Agents might **learn** about each other's rationality

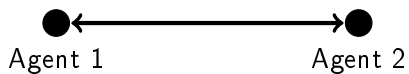
Example 2



Example 2

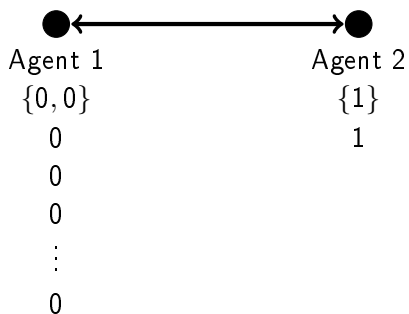


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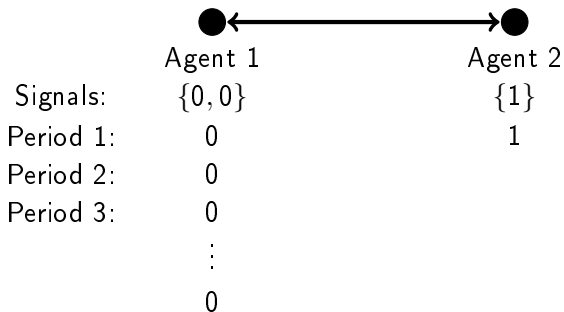


Signals:	{0, 0}	{1}
Period 1:	0	1

Example 2



Example 2




Assume $f(0) = 0$

Example 1 $\Rightarrow f(0)$ only impedes learning, though not causes disagreement per se.

Example 2 shows disagreement \Rightarrow setting $f(0) = 0$ makes our argument stronger

Example 2




	Agent 1	Agent 2
Signals:	$\{0, 0\}$	$\{1\}$
Period 1:	0	1
Period 2:	0	1 ← since $3p + q > 2$
Period 3:	0	
	\vdots	
	0	

Assume $f(0) = 0$, $3p + q > 2$

Agent 2: agent 1 is of level $k \geq 1 \Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1$

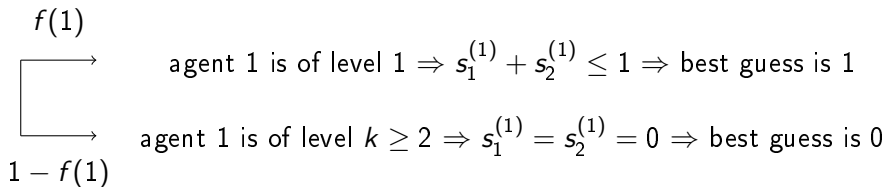
$$\mathbb{P} \left[\theta = 1 \mid s^{(2)} = 1, s_1^{(1)} + s_2^{(1)} \leq 1 \right] > \frac{1}{2} \Leftrightarrow 3p + q > 2$$

Example 2

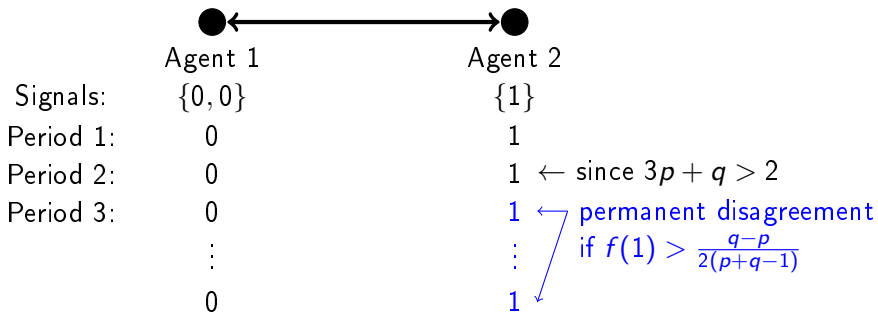


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Agent 2 knows:

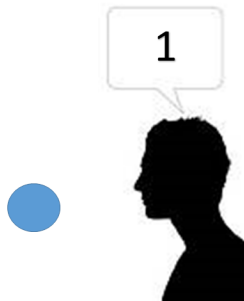
- ▶ level 1 agent 1 stops updating after period 1
- ▶ level $k \geq 2$ agent 1 stops updating after period 2
- ▶ \Rightarrow agent 1 stops updating after period 2

\Rightarrow agent 2 stops updating after period 3

Example 2



Agent 1

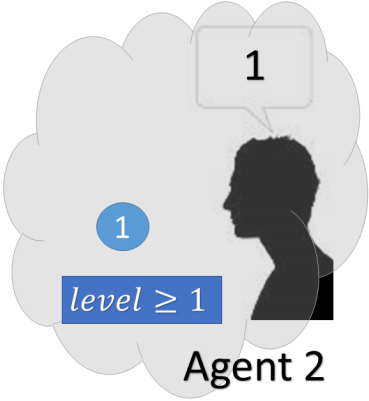


Agent 2

Example 2

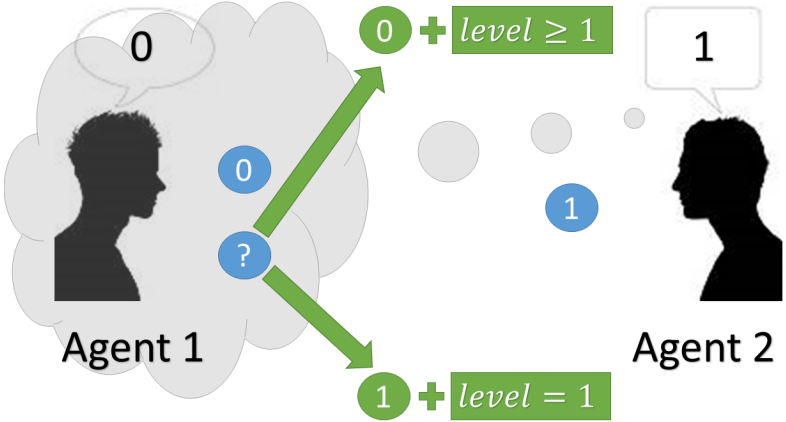


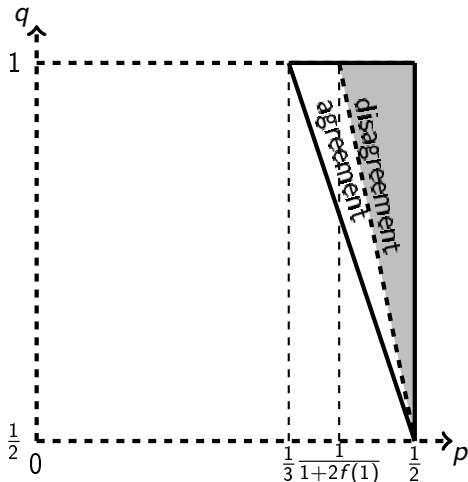
Agent 1



Agent 2

Example 2





Recall:

$$\mathbb{P}[\theta = 1] = p$$

$$\mathbb{P}[s = \theta \mid \theta] = q$$

Disagreement happens

1. if the **size of the doubt** $f(1)$ is sufficiently **high** (at least $f(1) > 0.5$)
2. **even if the cost of mistake is high**: if $p \approx 0.5$, then

$$\text{cost} = \underbrace{q}_{\text{correct belief about } \theta = 0} - \underbrace{0.5}_{\text{belief of indifference}}$$

Example 2 raises three questions:

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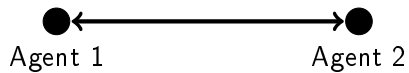
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1. Do we need to have **high doubt** about the other agent's rationality to get disagreement?
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3. Is it necessary for at least one agent to have **more than one signal**?

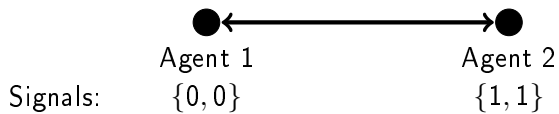
Example 2 raises three questions:

1. Do we need to have **high doubt** about the other agent's rationality to get disagreement? ← **NO (Example 3)**
2. Can we get disagreement when $f(1) = 0$? ← **YES (Example 4)**
3. Is it necessary for at least one agent to have **more than one signal**? ← **NO, but we need an incomplete network for it (Example 4)**

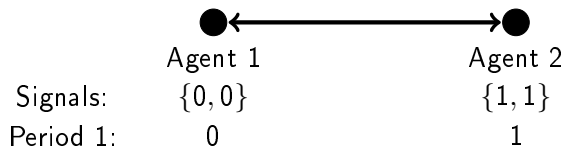
Example 3



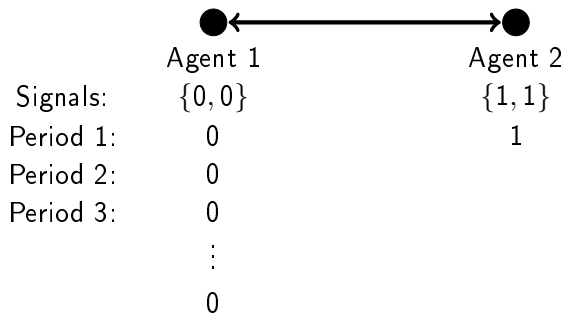
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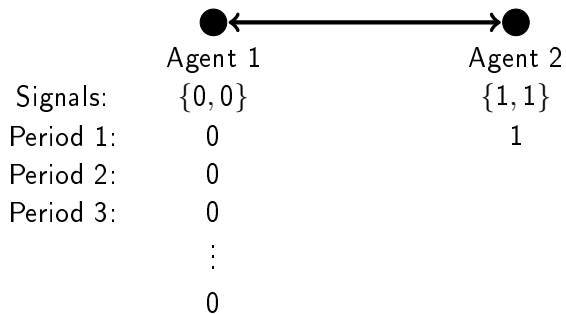
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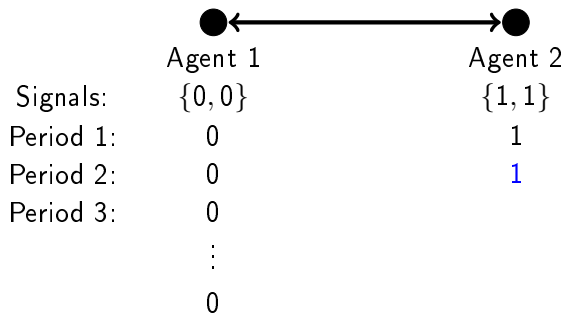


Example 3



Assume $f(0) = 0$

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Agent 2: agent 1 is of level $k \geq 1 \Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1$

$$\mathbb{P} \left[\theta = 1 \mid s_1^{(2)} = s_2^{(2)} = 1, s_1^{(1)} + s_2^{(1)} \leq 1 \right] > \frac{1}{2}$$

In contrast to Example 2, we don't need additional condition $3p + q > 2$ to guarantee 1

Example 3

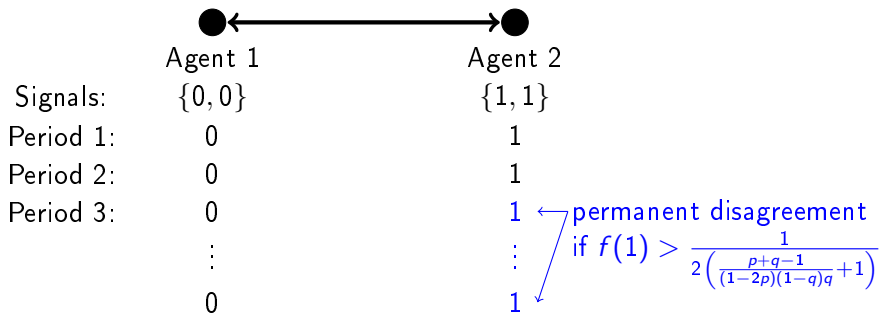


	Agent 1	Agent 2
Signals:	$\{0, 0\}$	$\{1, 1\}$
Period 1:	0	1
Period 2:	0	1
Period 3:	0	1 \leftarrow if $f(1) > \frac{1}{2\left(\frac{p+q-1}{(1-2p)(1-q)q} + 1\right)}$
	\vdots	
	0	

Assume $f(0) = 0$

$f(1)$
 $\left. \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \right\} \begin{array}{l} \text{agent 1 is of level 1} \Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1 \Rightarrow \text{best guess is 1} \\ \text{agent 1 is of level } k \geq 2 \Rightarrow s_1^{(1)} = s_2^{(1)} = 0 \Rightarrow \text{best guess is 0} \end{array}$
 $1 - f(1)$

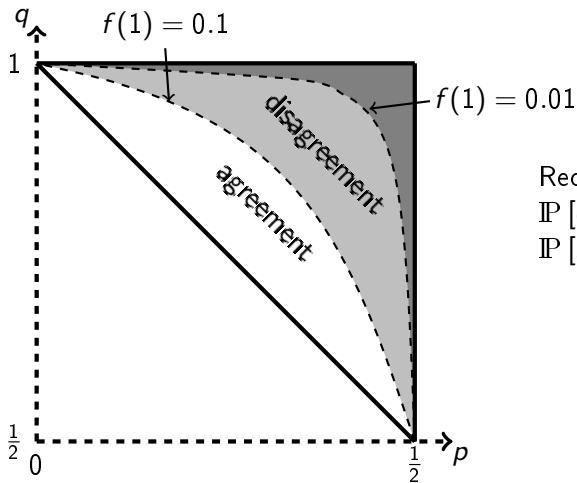
Example 3



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Agent 2 knows:

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 - ▶ \Rightarrow agent 1 stops updating after period 2
- \Rightarrow agent 2 stops updating after period 3



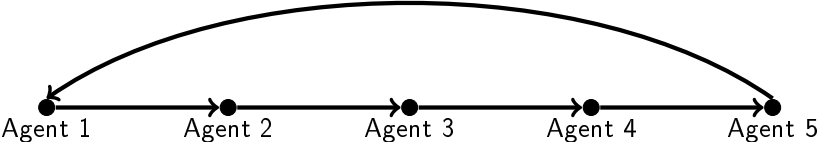
Recall:

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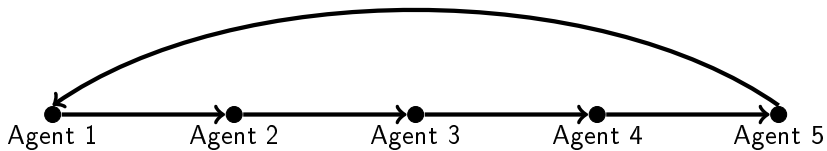
$$\mathbb{P}[s = \theta \mid \theta] = q$$

The size of the doubt $f(1)$ can be arbitrary small: for any $f(1) \in (0, 1)$ we can find $p \in (0, 0.5)$ and $q \in (0.5, 1)$ such that $p + q > 1$ and the agents permanently disagree with each other for some signals realizations.

Example 4

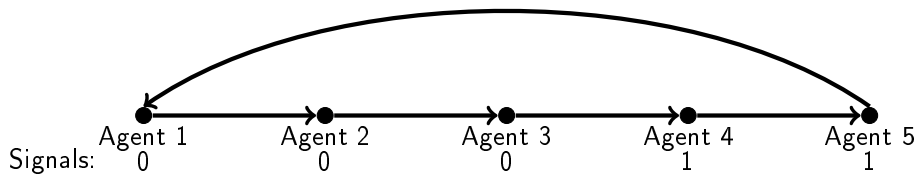


Example 4



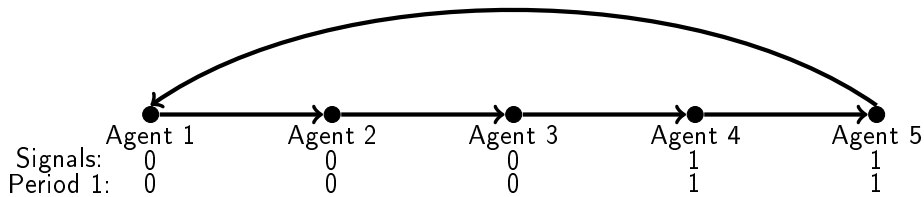
Assume $f(0) = f(1) = f(2) = 0$

Example 4



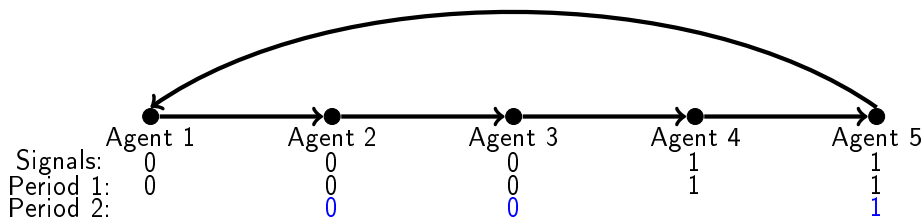
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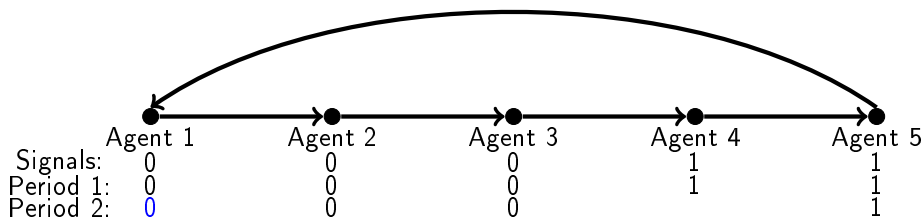
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agents 2, 3 and 5 infer that their neighbors got exactly the same signal as they did. \Rightarrow Their guesses do not change.

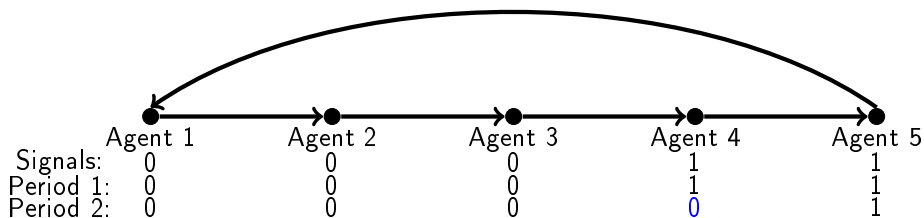
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agent 1 reasons that agent 5 got signal 1, but since the prior is biased towards 0, agent 1 still reports 0 in period 2.

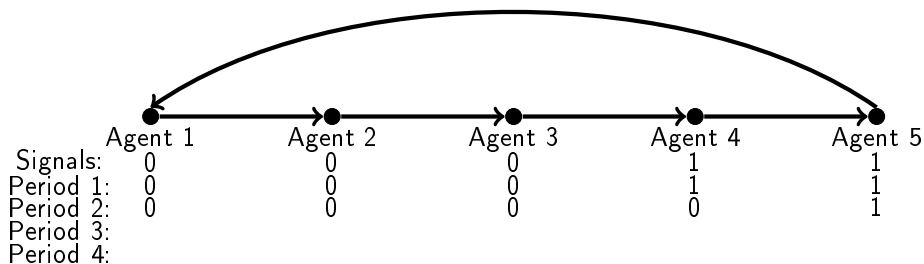
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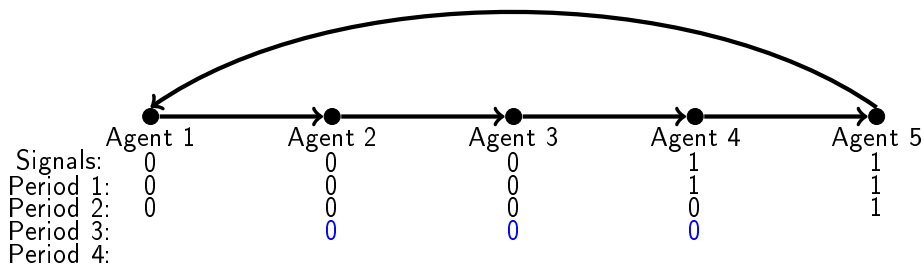
Agent 4 is in the reversed situation and therefore he is the only one who changes his guess.

Example 4



Assume $f(0) = f(1) = f(2) = 0 \Rightarrow$ It is common knowledge that everybody is behaving rationally in periods 1, 2 and 3.

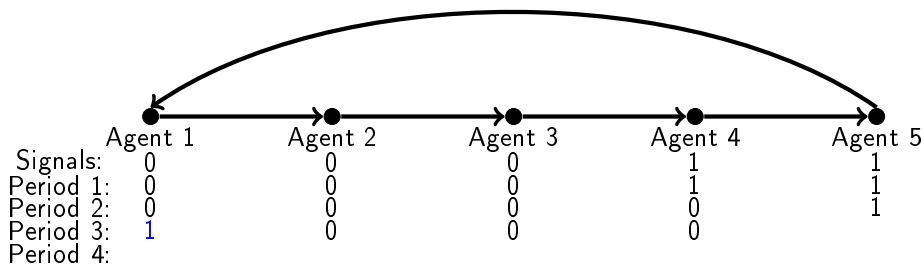
Example 4



Assume $f(0) = f(1) = f(2) = 0$

In period 3, agents 2, 3 and 4 observed only 0-s from their neighbors and therefore have no reason to change their guesses to 1.

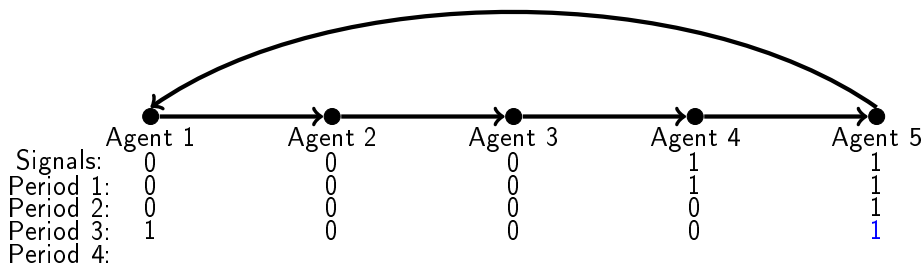
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agent 1 infers that both agents 4 and 5 got signals 1 (otherwise agent 5 would have switched to 0 in period 2). \Rightarrow Agent 1 reports 1 in period 3.

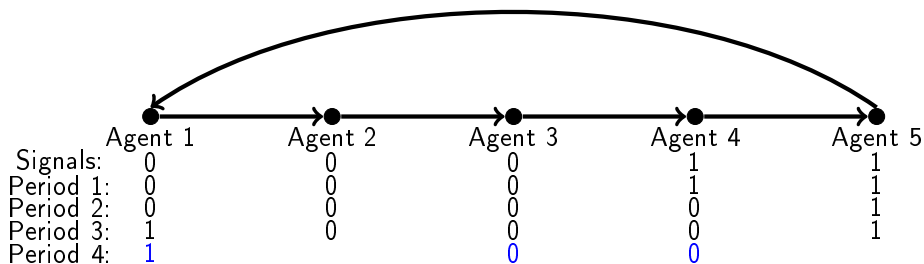
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agent 5 infers that agent 4 got 1 but agent 3 got 0. However, since agent 5 himself got 1, he would not switch to 0 in period 3.

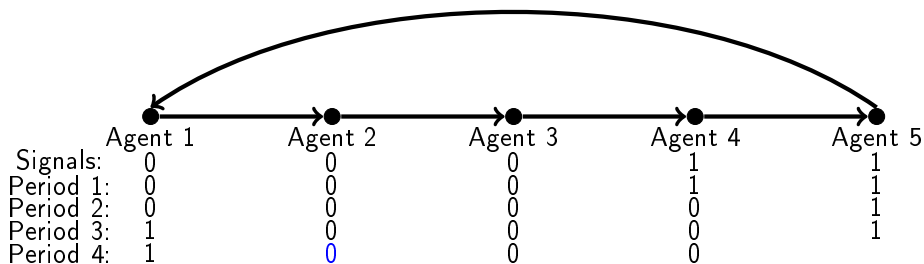
Example 4



Assume $f(0) = f(1) = f(2) = 0$

In period 4, agents 3 and 4 guess 0 as they received 0 from their neighbors. Similarly, agent 1 does not change his guess.

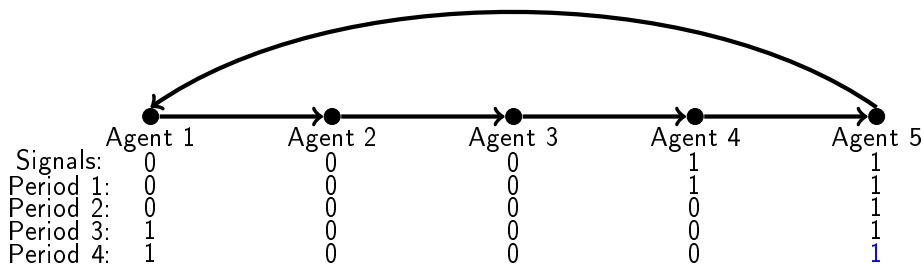
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agent 2 infers that agent 5 must have guessed 1 in periods 1 and 2 because that is the only situation when a rational agent would switch to 1 in period 3. That implies both agents 4 and 5 got 1. So, agent 2 now knows four signals: 0 (his own), 0 (received by agent 1), 1 (received by agent 5) and 1 (received by agent 4). Thus, he guesses 0 in period 4.

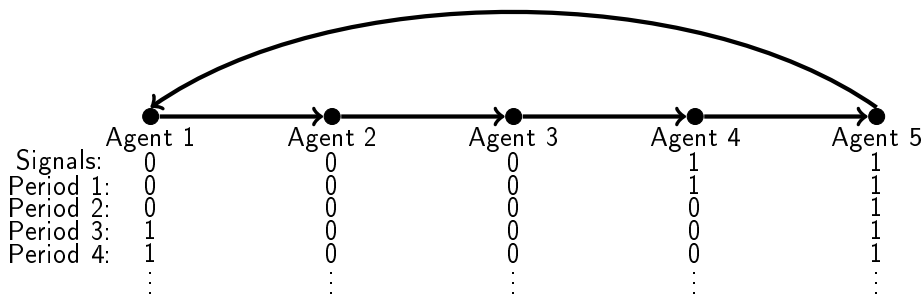
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agent 5 infers no new information from his neighbor' guess in period 2 since combination 1-0-1 simply impossible to hear from a rational agent. Thus, agent 5 guesses 1 in period 4.

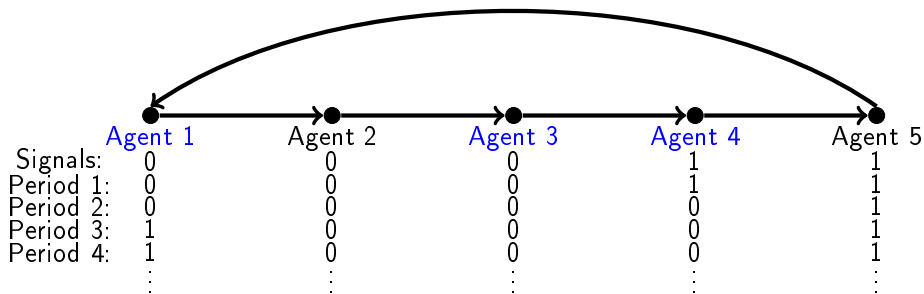
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

Example 4

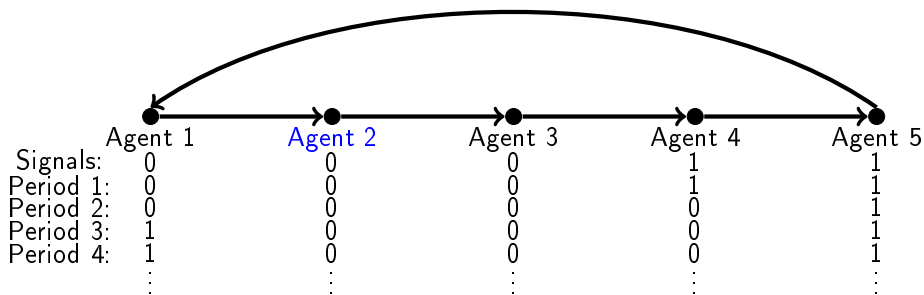


Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

- ▶ Agents 1, 3 and 4 have no reason to change their guesses since they agree with their neighbors.

Example 4



Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

- ▶ Agents 1, 3 and 4 have no reason to change their guesses since they agree with their neighbors.
- ▶ Agent 2 is correct, so his disagreement is “justified”.

Example 4

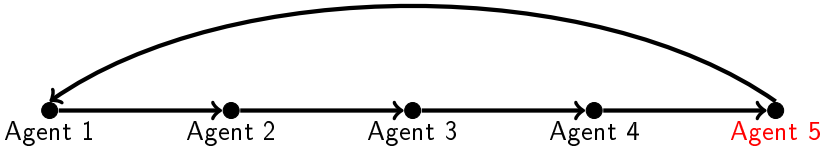


Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

- ▶ Agents 1, 3 and 4 have no reason to change their guesses since they agree with their neighbors.
- ▶ Agent 2 is correct, so his disagreement is “justified”.
- ▶ What is surprising in this example is the persistent disagreement of agent 5. (▶ skip proof ▶ middle step)

Example 4

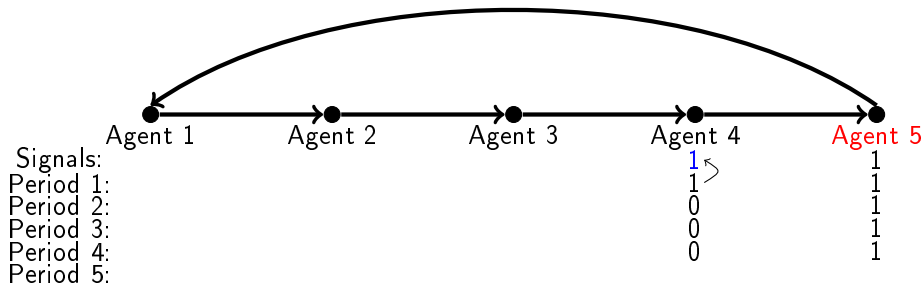


Signals:

- Period 1:
- Period 2:
- Period 3:
- Period 4:
- Period 5:

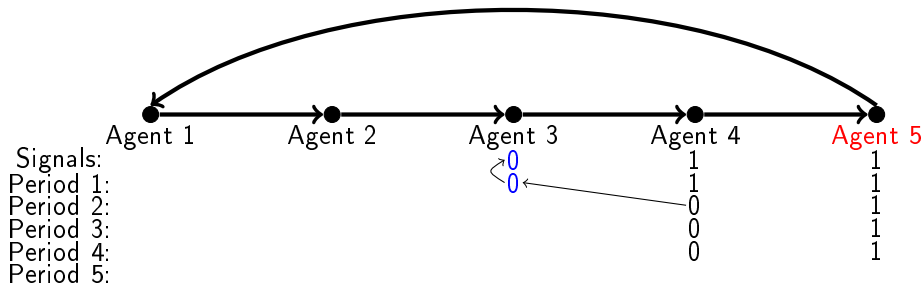
	1	1
	0	1
	0	1
	0	1

Example 4



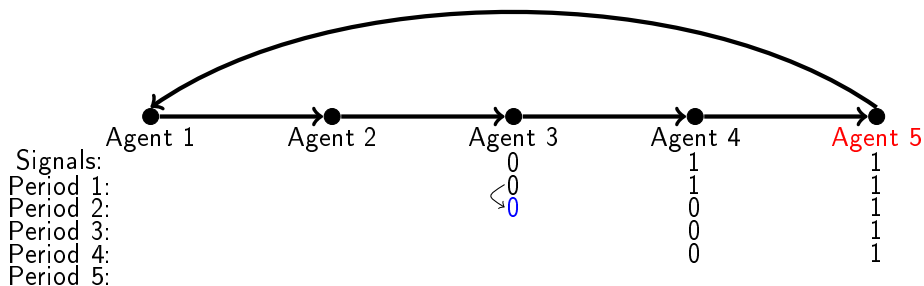
Agent 5 knows that agent 4 reports his signal in period 1. \Rightarrow Agent 4's signal is 1.

Example 4



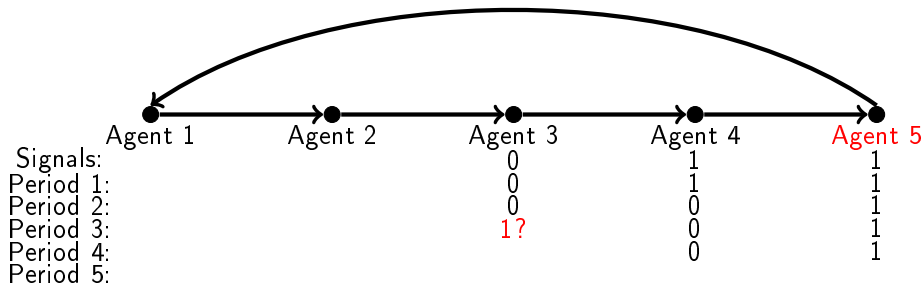
Agent 4's report 0 in period 2 indicates that agent 3 reported 0 in period 1 and therefore has signal 0.

Example 4



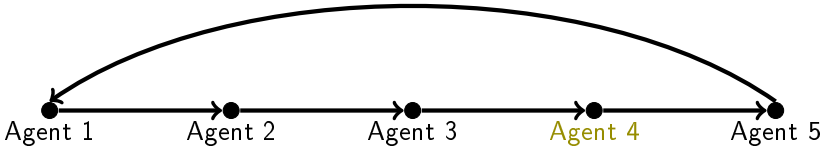
No agent of level $k \geq 3$ would ever report 1 in period 2 if he reported 0 in period 1. \Rightarrow Agent 3 reported 0 in period 2 as well.

Example 4



Suppose agent 3 reports 1 in period 3.

Example 4

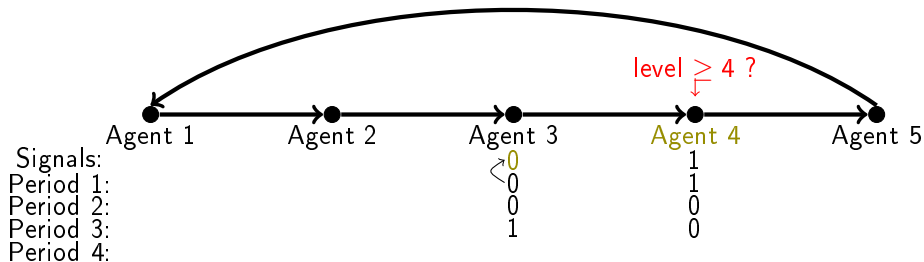


Signals:

- Period 1:
- Period 2:
- Period 3:
- Period 4:

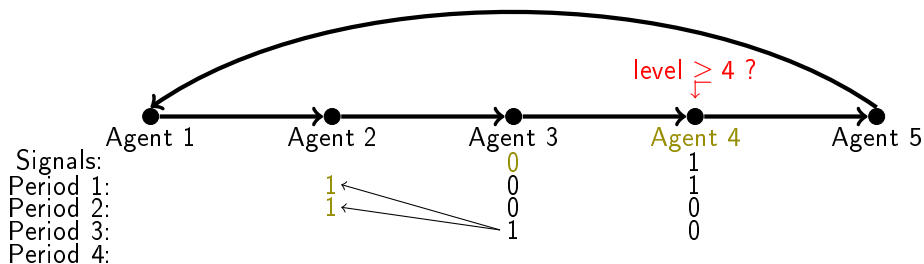
0	1
0	1
1	0
1	0

Example 4



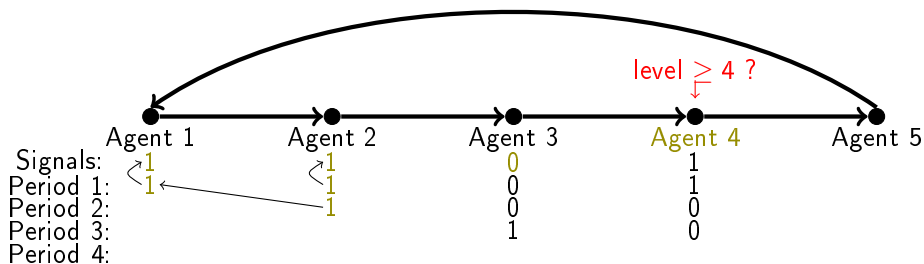
If agent 4 is of level 4 or higher, he is rational in period 4.

Example 4



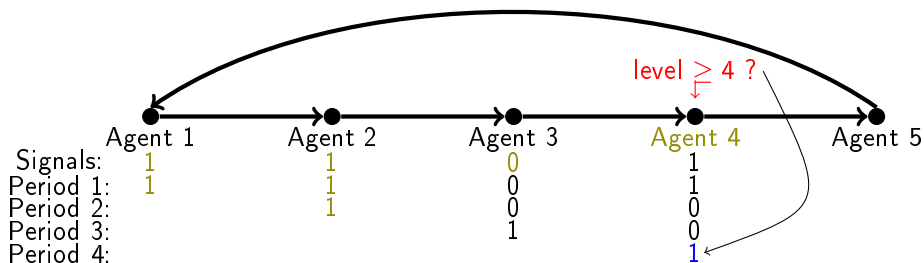
If agent 4 is of level 4 or higher, he is rational in period 4.

Example 4



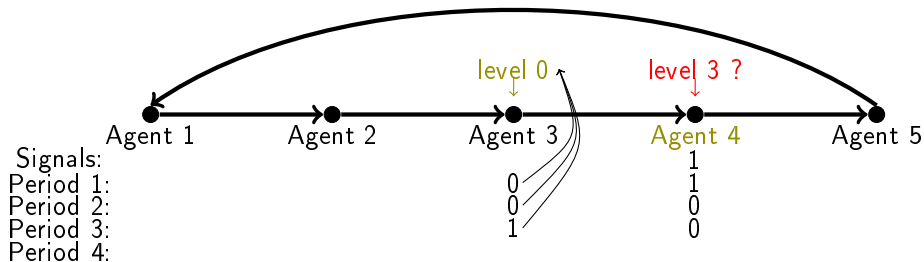
If agent 4 is of level 4 or higher, he is rational in period 4.

Example 4



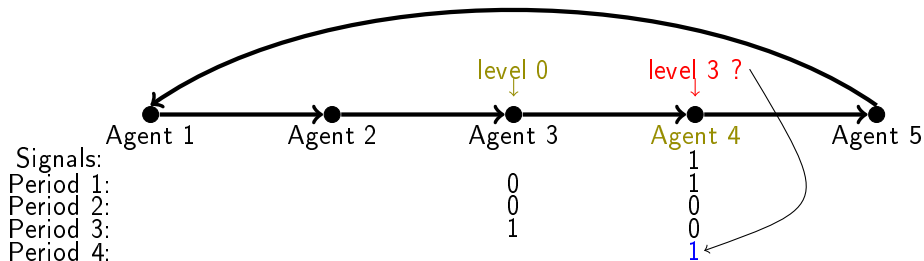
Agent 4 knows three signals: 0 (his own), 1 (agent 2), 1 (agent 1)
 \Rightarrow his best guess is 1

Example 4



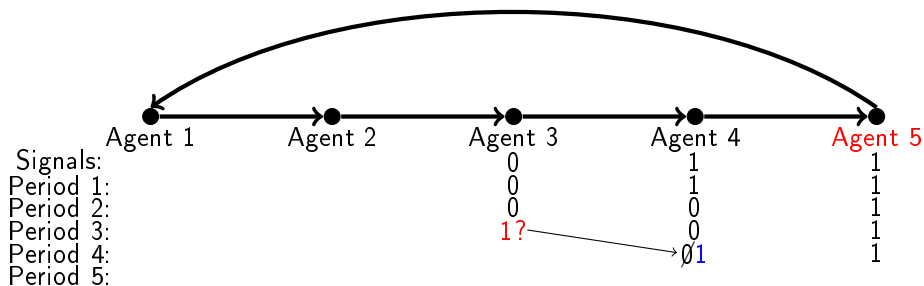
If agent 4 is of level 3, he believes that agent 3 is of level 0:
behavior 0-0-1 is inconsistent with neither level 2 nor with level 1

Example 4



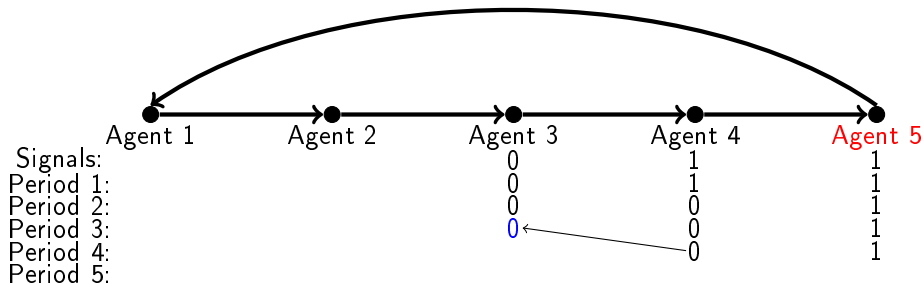
Agent 3 is of level 0, agent 4's own signal is 1 \Rightarrow best guess is 1

Example 4



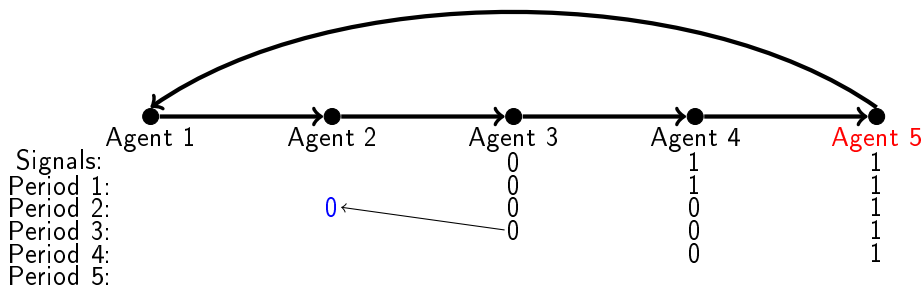
Suppose agent 3 reports 1 in period 3. \Rightarrow Agent 3 reports 1 in period 4. \Rightarrow Contradiction.

Example 4



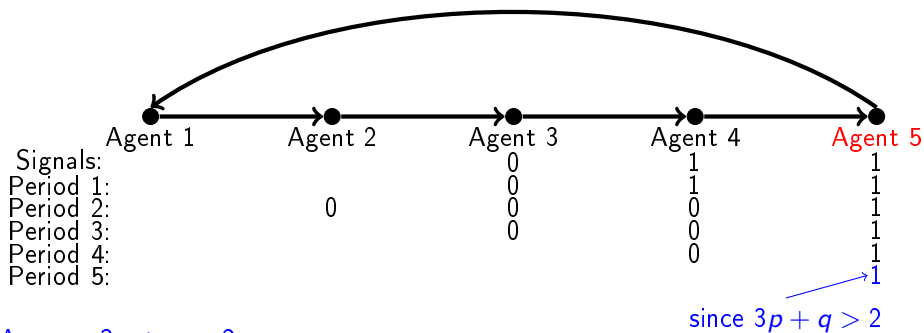
Suppose agent 3 reports 1 in period 3. \Rightarrow Agent 3 reports 1 in period 4. \Rightarrow Contradiction.

Example 4



Agent 3 is of level $\geq 3 \Rightarrow$ Agent 2 reports 0 in period 2

Example 4



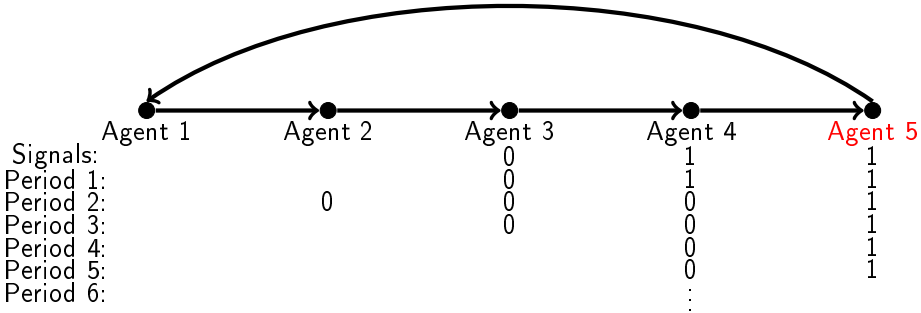
Assume $3p + q > 2$

That is all information agent 5 has by period 5. He knows that agent 3 has signal 0, agent 4 has signal 1 and agents 1 and 2 together have at least one 0 signal. So, agent 5's posterior belief is

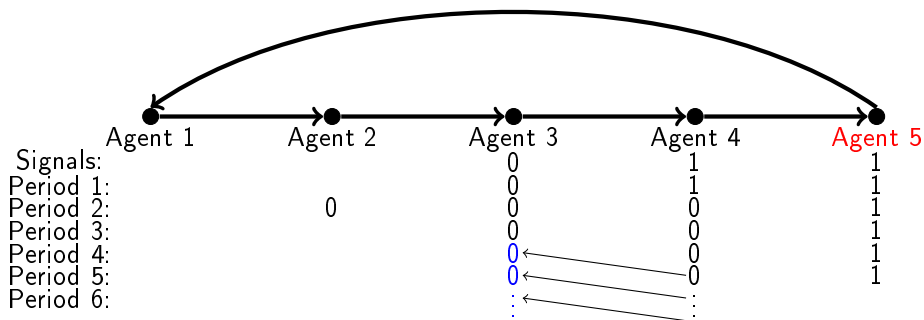
$$\mathbb{P} \left[\theta = 1 \mid s^{(1)} + s^{(2)} \leq 1, s^{(3)} = 0, s^{(4)} = s^{(5)} = 1 \right] > \frac{1}{2}$$

$$\Leftrightarrow 3p + q > 2$$

Example 4

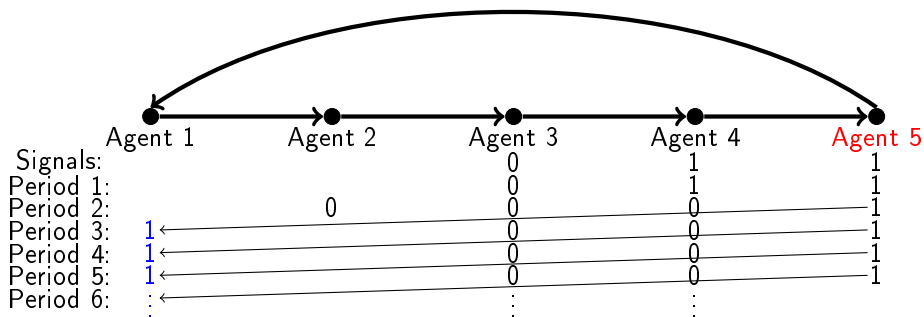


Example 4



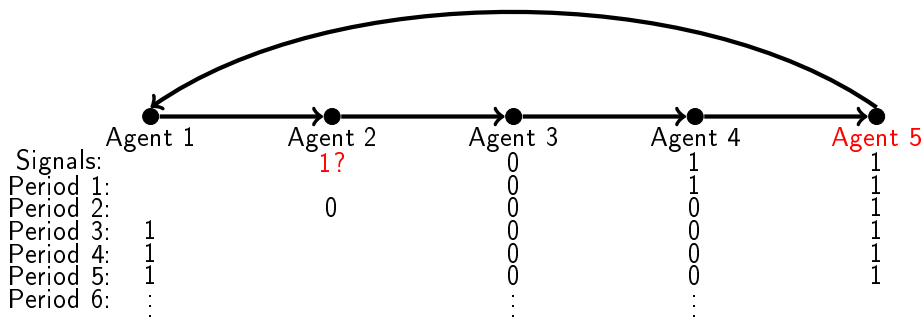
- Agent 3 must have reported all zeros in all previous periods:
agent 3 reports 1 in period $t - 2 \Rightarrow$ in period $t - 1$ agent 4 reasons
- ▶ either agent 3 is “smart” \Rightarrow agent 4 should “believe” him and report 1
 - ▶ or agent 3 is “stupid” \Rightarrow agent 4 should “ignore” him and report 1 (his own signal)

Example 4



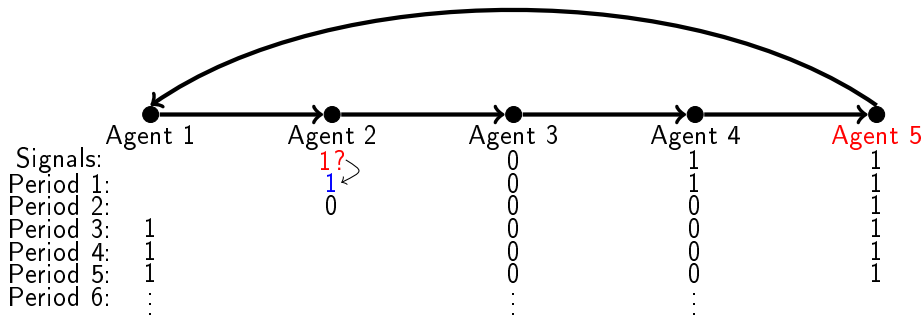
Agent 5 also knows that as long as he keeps reporting 1, agent 1 will guess 1 starting from period 3.

Example 4



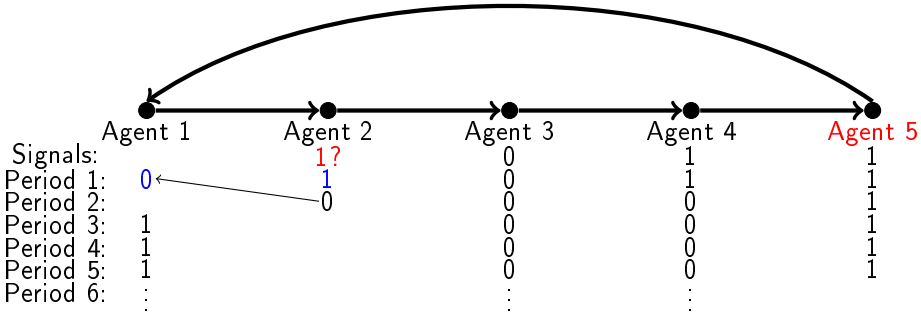
Suppose agent 2 received signal 1.

Example 4



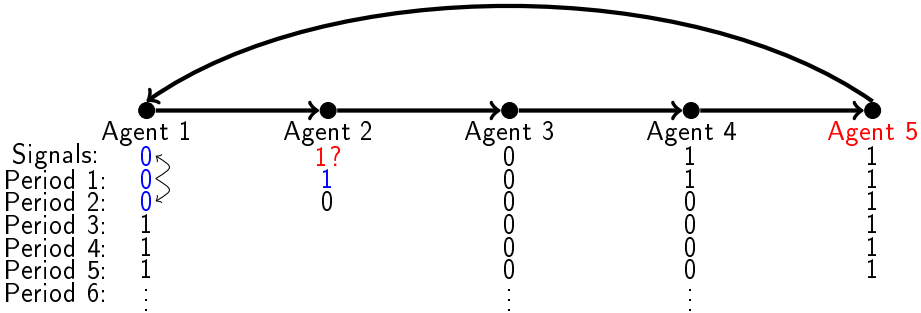
Suppose agent 2 received signal 1.

Example 4



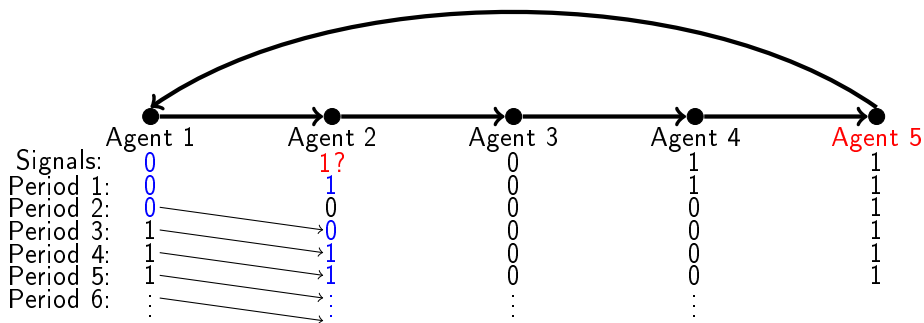
Suppose agent 2 received signal 1.

Example 4



Suppose agent 2 received signal 1.

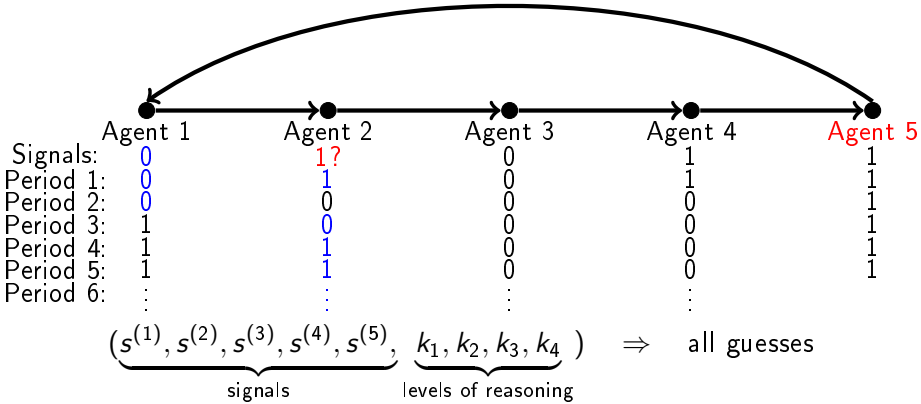
Example 4



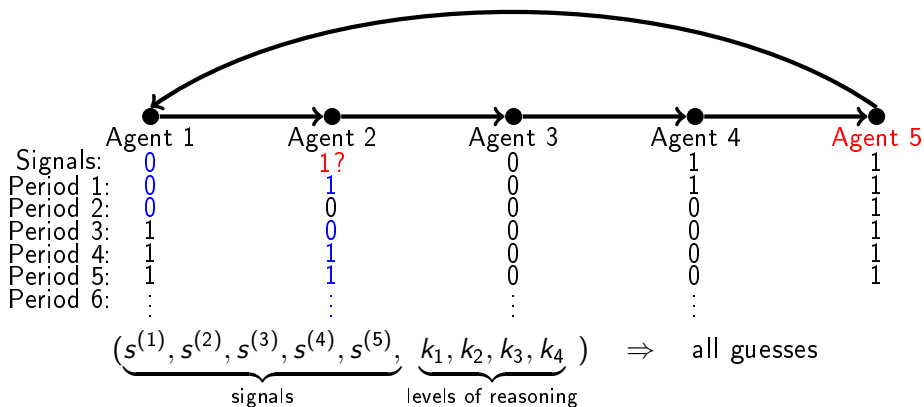
Suppose agent 2 received signal 1.

Similar to agent 4, agent 2 repeats the guess of his neighbor starting from period 3.

Example 4



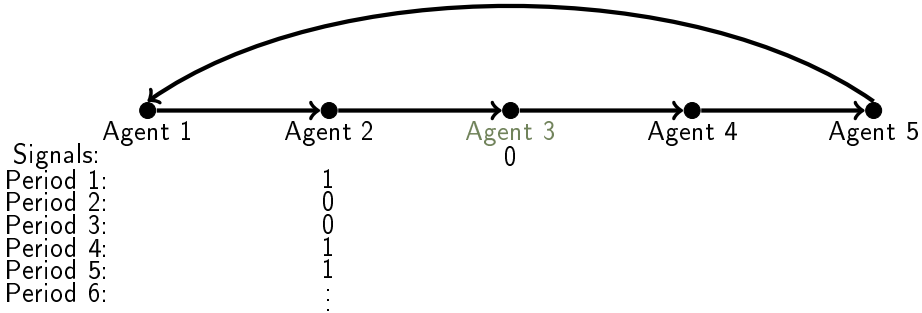
Example 4



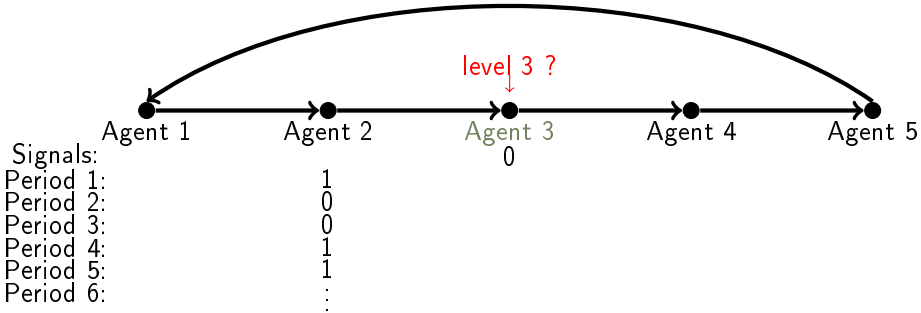
Suppose this configuration is correct.

- ▶ signals are fixed
- ▶ agent 5 cannot exclude any level $k_i \geq 3$ of reasoning for agents 1, 2 and 4
- ▶ what about agent 3?

Example 4

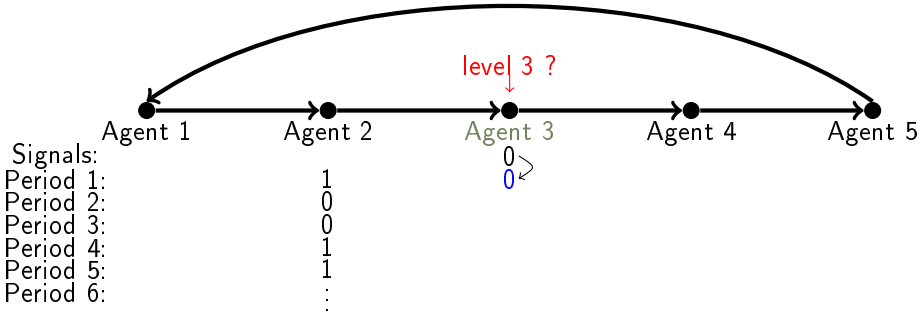


Example 4



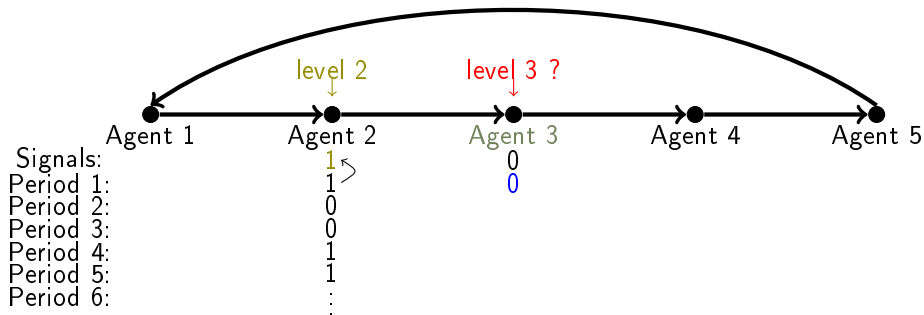
Suppose agent 3 is of level 3.

Example 4



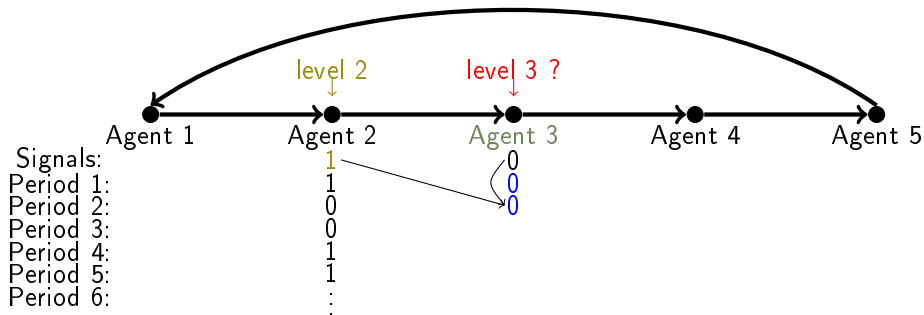
In period 1, agent 3 reports 0, his signal.

Example 4



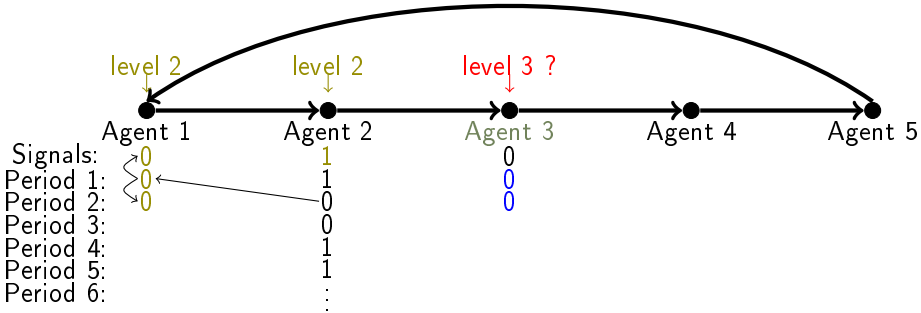
In period 2, thinking that agent 2 is of level 2, agent 3 infers that agent 2 has 1

Example 4



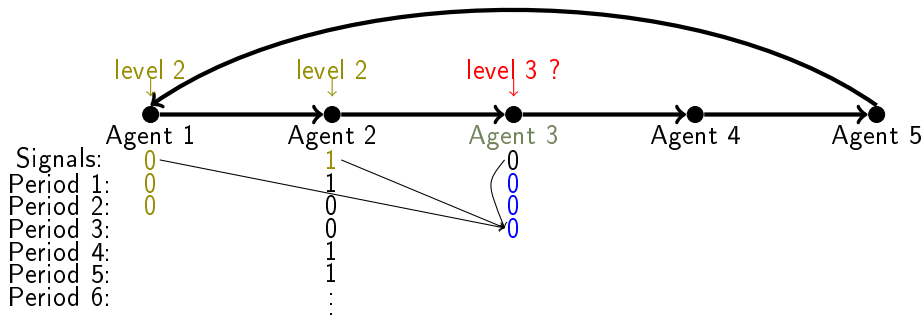
In period 2, thinking that agent 2 is of level 2, agent 3 infers that agent 2 has 1 but it is still optimal to report 0 since the prior is biased towards 0.

Example 4



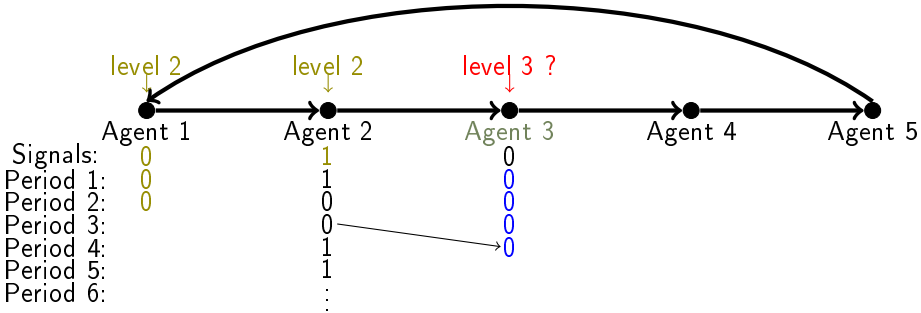
In period 3, agent 3 infers that agent 1 has signal 0

Example 4



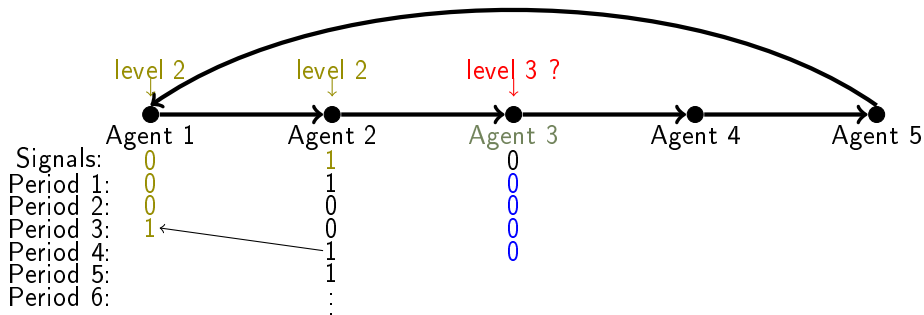
In period 3, agent 3 infers that agent 1 has signal 0, so he reports 0 again.

Example 4



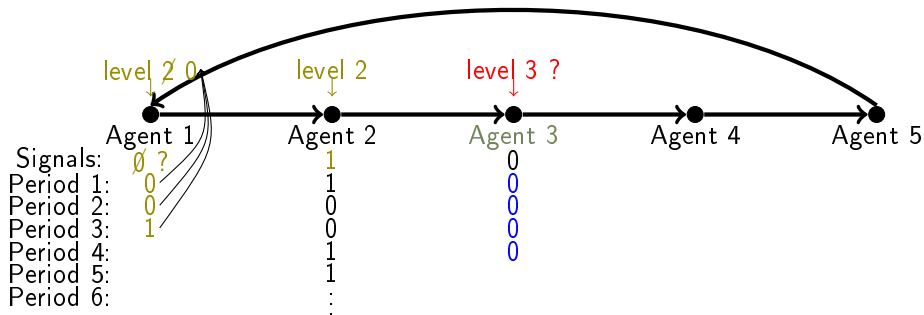
In period 4, agent 4 gets no reason to change the guess.

Example 4



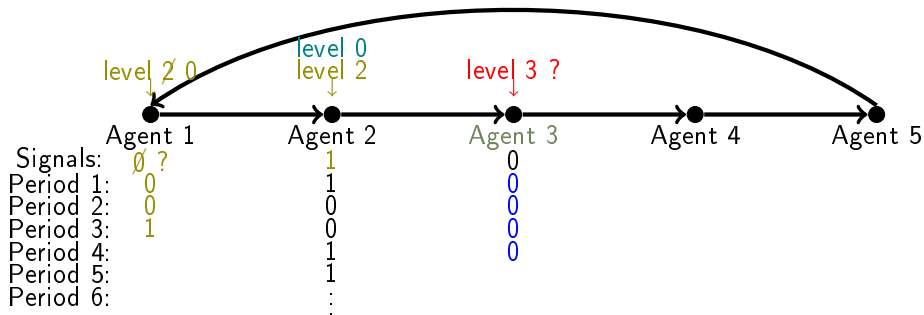
If agent 2 is of level 2, then agent 1 reported 1 in period 3

Example 4



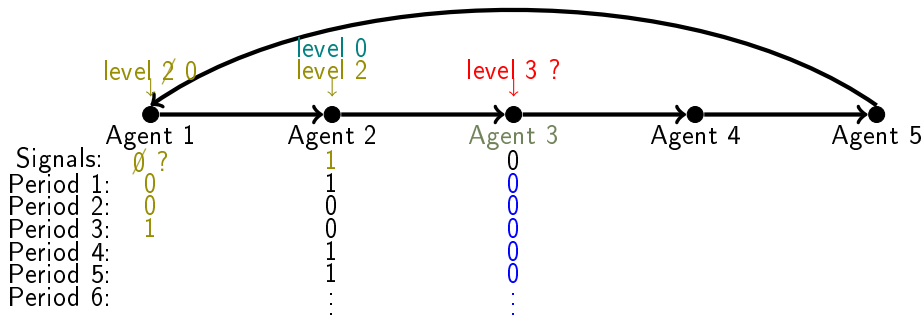
If agent 2 is of level 2, then agent 1 reported 1 in period 3 \Rightarrow
Agent 1 must be of level 0

Example 4



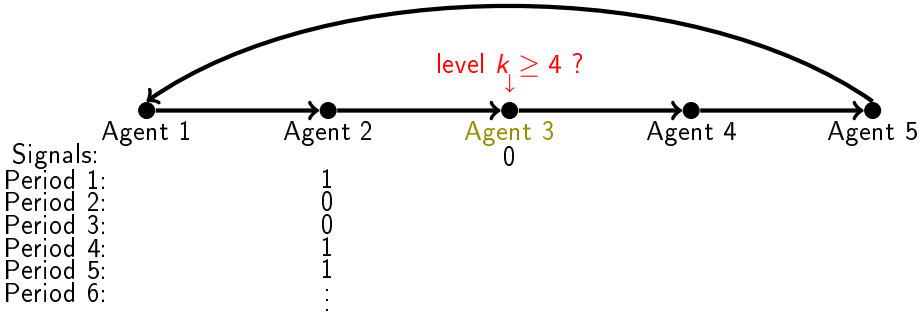
If agent 2 is of level 2, then agent 1 reported 1 in period 3 \Rightarrow
Agent 1 must be of level 0 or agent 2 is of level 0.

Example 4



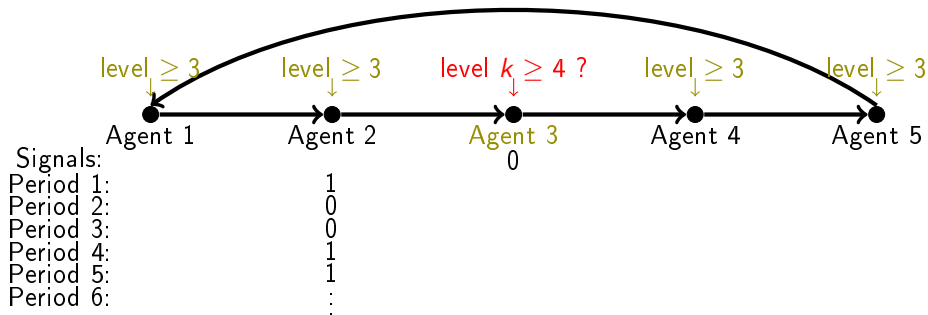
If agent 2 is of level 2, then agent 1 reported 1 in period 3 \Rightarrow
Agent 1 must be of level 0 or agent 2 is of level 0. Either way, the
best guess is 0.

Example 4



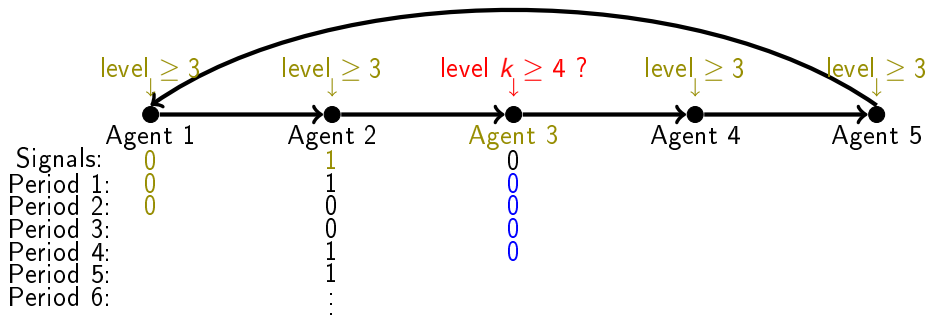
Suppose agent 3 is of level $k \geq 4$.

Example 4



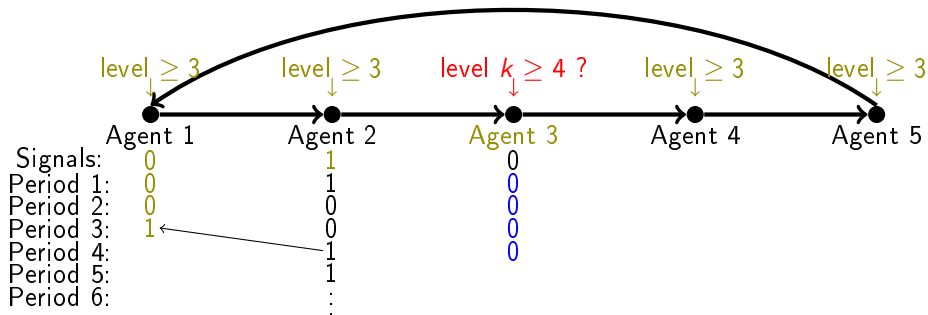
Agent 3 thinks that everybody else is of level at least 3.

Example 4



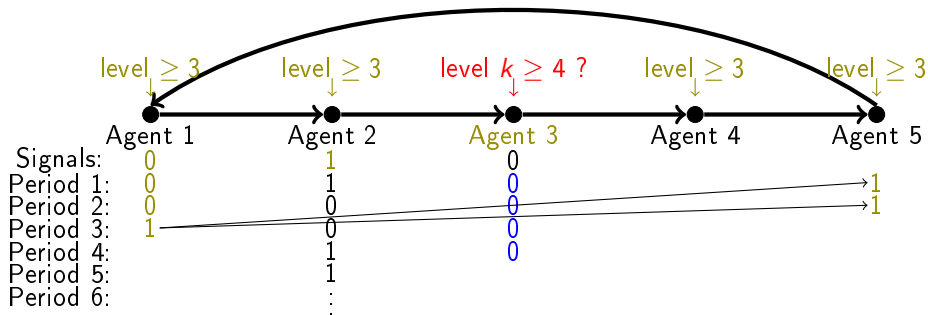
Up to period 4, level $k \geq 4$ agent 3 reasons exactly the same as level 3 agent 3.

Example 4



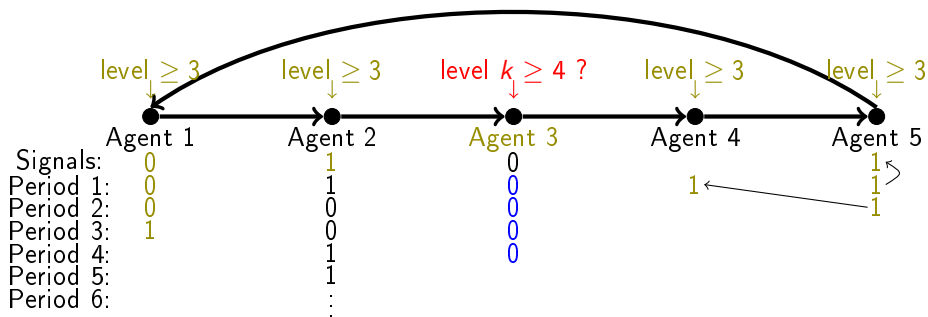
In period 5, agent 3 concludes that agent 1 reported 1 in period 3 (since agent 2 is of level 3 or higher)

Example 4



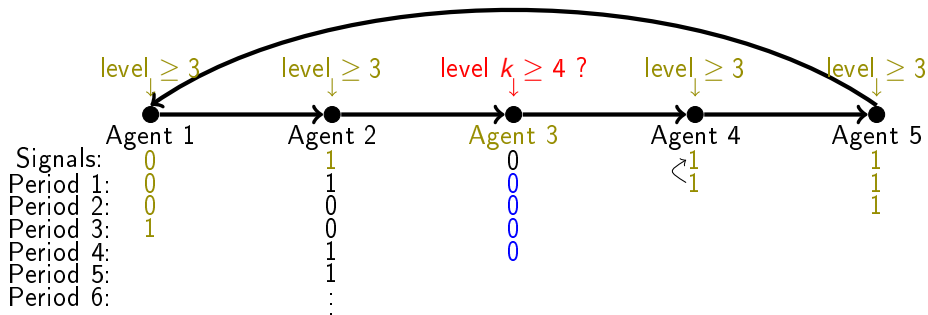
In contrast to level 3 agent 3, level $k \geq 4$ agent 3 makes the right conclusion in period 5: agent 1 reports 0-0-1 because agent 5 reports 1-1.

Example 4



Being of level at least 3, agent 5 reports 1-1 if and only if he gets signal 1 and observes 1 from agent 4.

Example 4

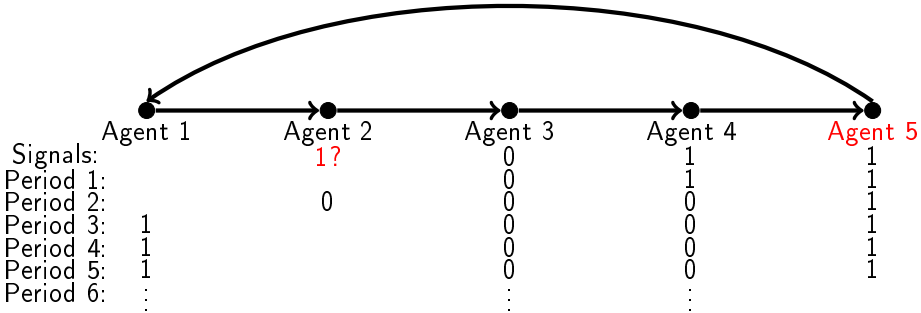


Since agent 4 follows his own signal in period 1, agent 3 concludes that agent 4 has signal 1.

Example 4

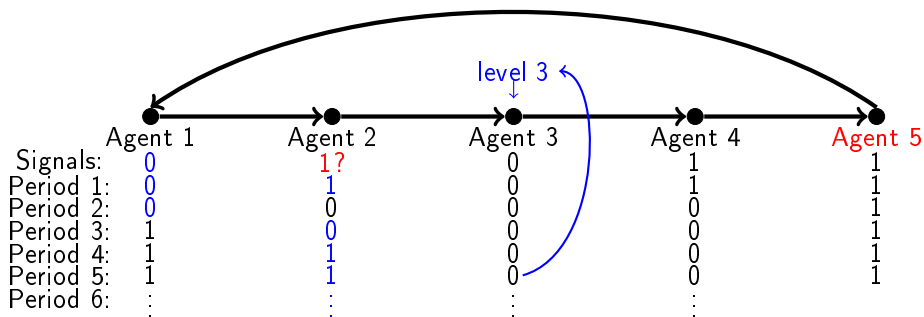


Example 4



Assume agent 2 got signal 1.

Example 4

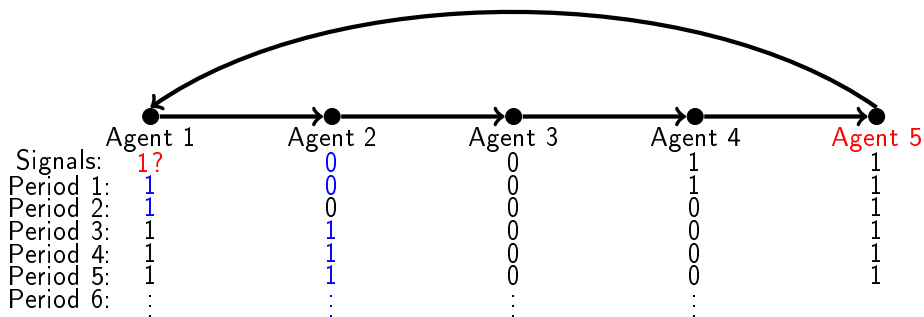


Assume agent 2 got signal 1.

Period 6: Agent 5 concludes that agent 3 reported all 0s in periods 1-4
⇒ Consistent with any level $k \geq 3$ of reasoning for agent 3

Period 7: Agent 5 concludes that agent 3 reported all 0s in periods 1-5
⇒ Consistent only with level 3 behavior for agent 3

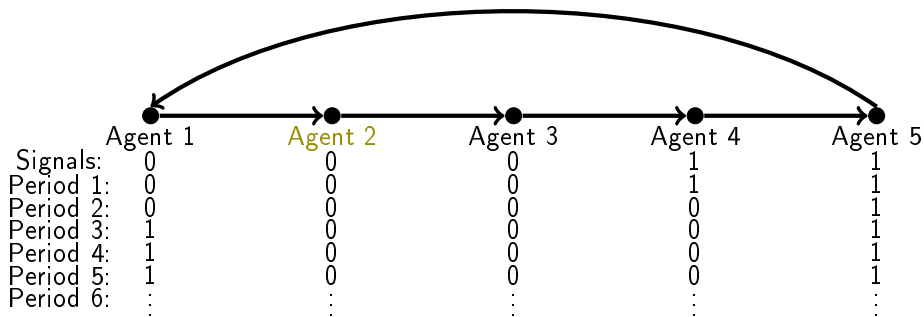
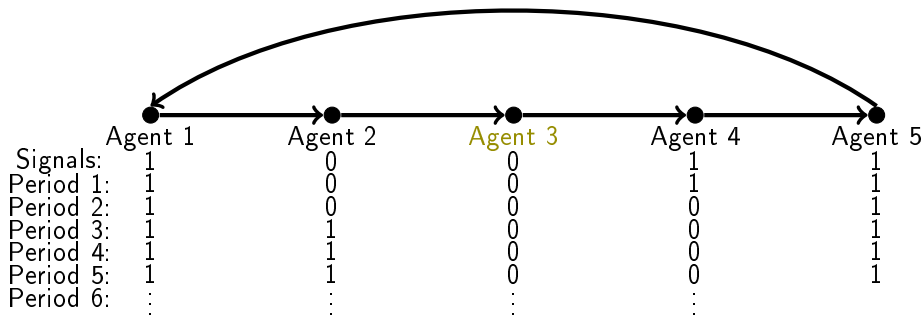
Example 4



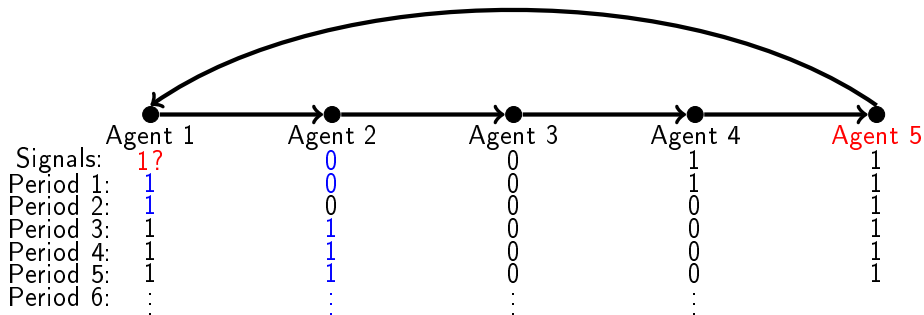
Assume agent 1 got signal 1.

Configuration is consistent with any level $k \geq 3$ of reasoning for agents 1, 2 and 4.

What about agent 3?



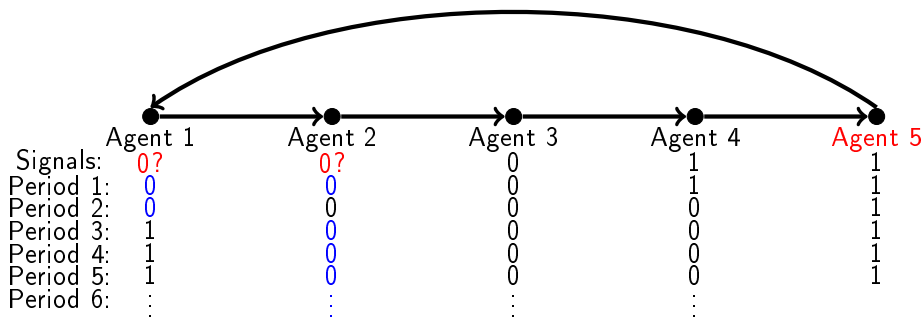
Example 4



Assume agent 1 got signal 1.

⇒ Consistent with any level $k \geq 3$ of reasoning for agents 1, 2, 3 and 4

Example 4

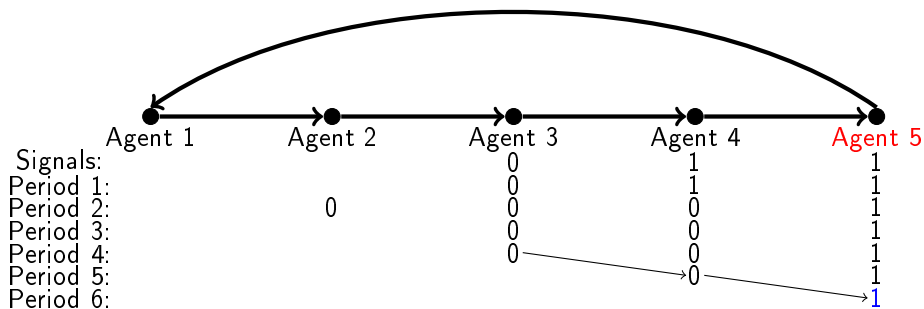


Assume both agents 1 and 2 received signals 0.

This is in fact the situation that is realized (though agent 5 does not know it).

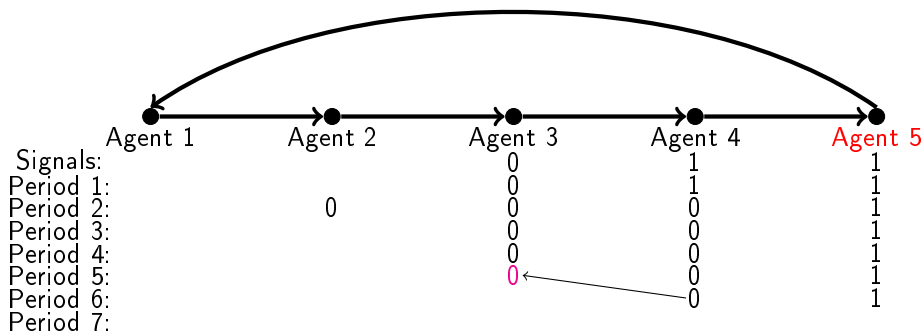
⇒ Consistent with any level $k \geq 3$ of reasoning for agents 1, 2, 3 and 4

Example 4



In period 6, agent 5 does not receive any new information (he anticipates agent 4 to report 4) \Rightarrow agent 5 does not change his guess

Example 4



In period 7, agent 5 excludes configuration

$$s^{(1)} = 0, s^{(2)} = 1, s^{(3)} = 0, s^{(4)} = 1, s^{(5)} = 1, k_3 \geq 4$$

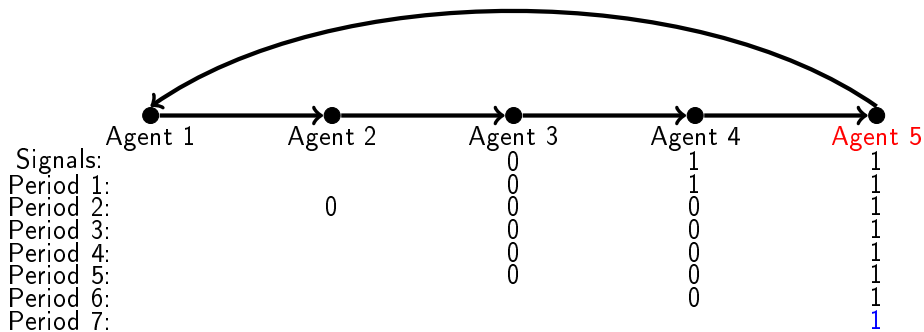
$f(3)$

agent 3 is of level 3 $\Rightarrow s^{(1)} + s^{(1)} \leq 1 \Rightarrow$ best guess is 1

agent 3 is of level $k \geq 4 \Rightarrow s_2^{(1)} = 0 \Rightarrow$ best guess is 0

$1 - f(3)$

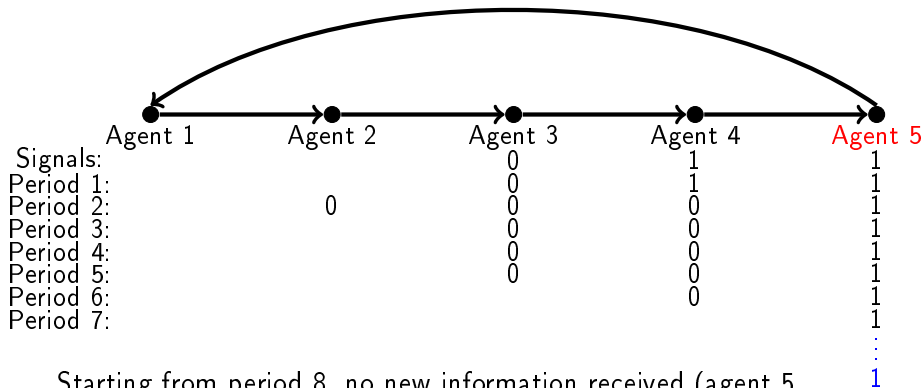
Example 4



In period 7, agent 5 guesses 1 if

$$f(3) > \frac{1 - 2p}{p + q - 1}$$

Example 4



Starting from period 8, no new information received (agent 5 anticipates receiving 0 from his neighbor) \Rightarrow disagreement is permanent

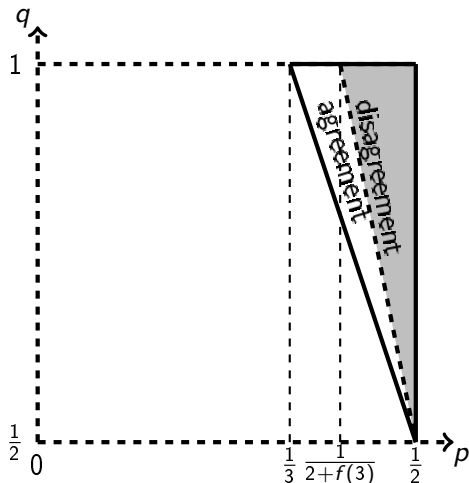
Example 4



No matter how much time has passed, **agent 5 cannot completely exclude** the situation when **agent 3 is of level 3** and therefore cannot exclude the situation when agent 2 has signal 1. \Rightarrow **Agent 5 does not know that agent 2 disagrees with him.**

Agent 2 does know that agent 5 disagrees with him but he **does not know why** this happens: it's because agent 5 thinks that agent 3 might be of level 3, or it's because agent 3 has signal 1.

Example 4



Recall:

$$\mathbb{P}[\theta = 1] = p$$

$$\mathbb{P}[s = \theta \mid \theta] = q$$

We assumed $f(0) = f(1) = f(2) = 0$

Disagreement can happen even for the smallest doubt $f(3)$!

Conclusion

Today: Examples

- ▶ Cognitive hierarchical model in networks
- ▶ Small doubt ^{sometimes} \Rightarrow permanent disagreement (\Rightarrow trade)
- ▶ Idea: Agent who disagrees can never learn which of several scenarios is realized (learning stops in finite time).

Future research: General results