Cognitive Hierarchical Model in Networks

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Paradox

- Aumann (1976): it is impossible to agree to disagree $\Rightarrow$ No trade theorem
- Real world proves otherwise: we disagree and we trade

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Idea

Deviate from Aumann just a little bit $\Rightarrow$ disagreement is possible

Very preliminary, your comments are very welcome!

Agreeing to Disagree

Aumann (1976)

If two individuals share the same prior and have common knowledge among them of

- their information partitions
- their rationality (that is, both agents use Bayes rule to update their beliefs)
- their posteriors

then these posteriors must be the same (they cannot agree to disagree)

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Geanakoplos and Polemarchakis (1982): communication of posteriors

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Cave (1983), Bacharach (1985): communication of decisions among like-minded agents

Gale and Kariv (2003): communication of decisions in a sufficiently connected network

Existing literature: Agreement result is robust to relaxing

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Ellison and Fudenberg (1993), DeMarzo, Vayanos, and Zwiebel (2003), Golub and Jackson (2010), Molavi, Tahbaz-Salehi and Jadbabaie (2017) etc : agreement for boundedly rational agents

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This paper: but not to *common knowledge* of rationality.

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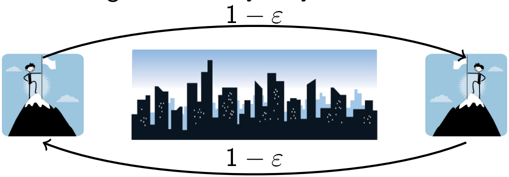
Intuition: in the spirit of **Rubinstein (1989) Email Game** who shows that "the game-theoretic prediction for the almost common knowledge situation is very different from the situation with common knowledge"

Coordination game: victory only if both armies attack



Eventually, the message is lost with probability $1-(1-\varepsilon)^t \underset{t\to\infty}{\to} 1$

Once the message is not received, it is not optimal to attack:

$$\underbrace{\varepsilon}_{\text{initial massage is lost} \Rightarrow \text{other army does not attack}} > \underbrace{(1-\varepsilon)\varepsilon}_{\text{returned massage is lost}}$$

Outline

Model

State and signals

- state: $\theta \in \{0, 1\}$
- common prior: $\mathbb{P}[\theta = 1] = p \in (0, 0.5)$

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- common prior: $\mathbb{P}[\theta = 1] = p \in (0, 0.5)$
- private signals: independent conditional on θ

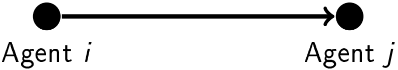
$$s \in \{0,1\} \quad \mathbb{P}[s = \theta \mid \theta] = q \in (0.5, 1)$$

- assume $\mathbb{P}[\theta = s \mid s] > 0.5$, or equivalently $p + q > 1$

Model

Communication game

- directed network:



  means Agent j *listens to* Agent i
- at each period $t \geq 1$, each agent guesses θ sincerely
  - assumption: no strategic considerations
- if agent j listens to agent i, then agent j observes all guesses of agent i from all **past** periods

Model

Belief formation

Based on the cognitive hierarchical (CH) theory proposed in Camerer et al (2004)

Level 0 agent randomizes each period: $\begin{cases} 0, & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$

Level 1 agent thinks that all other players are of level 0

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Parameter of the model: function f — probability distribution over the levels of thinking

$$f(k) \in [0,1], \quad \sum_{k=0}^{+\infty} f(k) = 1$$

Level $k \geq 2$ agent thinks that the levels of thinking of all other players are independently distributed over 0 through $k-1$ according to the distribution f truncated at level $k-1$.

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Novelty: f might be inconsistent with realized distribution. We assume everybody is of level ∞

Model

Discussion

- Two levels of uncertainty: state of the world and realized levels of thinking.
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- Suppose $f(+\infty) = 1 - \varepsilon$ for $\varepsilon > 0$ very small
- *Is it possible that agents not only fail to learn that everybody is of level ∞, but also permanently disagree on the state?*

Outline

Model

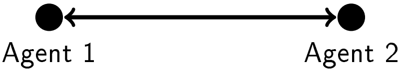
Example 1

Example 2

Example 3

Example 4

Example 1



Example 1

Agent 1 — Agent 2

Signals: 0 — 1

Example 1

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1

Recall: $p + q > 1 \Rightarrow$ always optimal to guess the signal in period 1: $\mathbb{P}[\theta = s \mid s] > 0.5$

Example 1

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	
	⋮	
	0	

Recall: prior is biased towards 0: $\mathbb{P}[\theta = 1] = p < 0.5 \Rightarrow$ agent 1 always reports 0

Example 1

	Agent 1	Agent 2
Signals:	0	1
Period 1:	0	1
Period 2:	0	?
	⋮	
	0	

Agent 2:

$f(0)$ → agent 1 is of level 0 $\Rightarrow$ best guess is 1

$1 - f(0)$ → agent 1 is of level $k \geq 1$ $\Rightarrow$ best guess is 0

Example 1

	Agent 1	Agent 2	
Signals:	0	1	
Period 1:	0	1	
Period 2:	0	?	$\leftarrow$ depends on $f(0)$
	$\vdots$		
	0		

Agent 2:

$f(0)$ $\longrightarrow$ agent 1 is of level 0 $\Rightarrow$ best guess is 1

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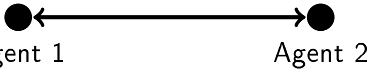
Example 1



	Agent 1	Agent 2	
Signals:	0	1	
Period 1:	0	1	
Period 2:	0	?	$\leftarrow$ depends on $f(0)$
	$\vdots$	$\vdots$	
	0	0	$\leftarrow$ agreement is inevitable

$$\mathbb{P}\left[s^{(2)} = 1,\ y_1^{(1)} = \ldots = y_{t-1}^{(1)} = 0 \mid \text{agent 1 is of level } 0\right] \underset{t\to\infty}{\longrightarrow} 0$$

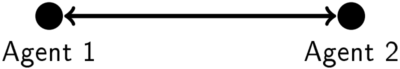
$\Rightarrow$ eventually, agent 2 learns that agent 1 is of level $k \geq 1$

Example 1

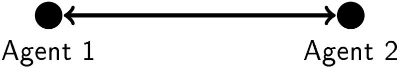
	Agent 1	Agent 2	
Signals:	0	1	
Period 1:	0	1	
Period 2:	0	?	$\leftarrow$ depends on $f(0)$
	$\vdots$	$\vdots$	
	0	0	$\leftarrow$ agreement is inevitable

1. Absence of c.k. of rationality does not necessarily leads to disagreement
2. Agents might learn about each other's rationality

Example 2



Example 2



Signals: $\{0, 0\}$ $\{1\}$

Example 2

	Agent 1	Agent 2
Signals:	$\{0, 0\}$	$\{1\}$
Period 1:	0	1

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Period 2:	0	
Period 3:	0	
	$\vdots$	
	0	

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	Agent 1	Agent 2
Signals:	$\{0, 0\}$	$\{1\}$
Period 1:	0	1
Period 2:	0	
Period 3:	0	
	$\vdots$	
	0	

Assume $f(0) = 0$

Example 1 $\Rightarrow$ f(0) only impends learning, though not causes disagreement per se.
Example 2 shows disagreement $\Rightarrow$ setting $f(0) = 0$ makes our argument stronger

Example 2

	Agent 1	Agent 2
Signals:	$\{0,0\}$	$\{1\}$
Period 1:	0	1
Period 2:	0	1 $\leftarrow$ since $3p+q>2$
Period 3:	0	
	$\vdots$	
	0	

Assume $f(0)=0$, $3p+q>2$

Agent 2: agent 1 is of level $k \geq 1 \Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1$

$$\mathbb{P}\left[\theta = 1 \mid s^{(2)} = 1,\ s_1^{(1)} + s_2^{(1)} \leq 1\right] > \frac{1}{2} \quad \Leftrightarrow \quad 3p+q>2$$

Example 2

	Agent 1	Agent 2	
Signals:	$\{0, 0\}$	$\{1\}$	
Period 1:	0	1	
Period 2:	0	1	$\leftarrow$ since $3p + q > 2$
Period 3:	0	1	$\longleftarrow$ if $f(1) > \frac{q-p}{2(p+q-1)}$
	$\vdots$		
	0		

Assume $f(0) = 0$, $3p + q > 2$

$f(1)$: agent 1 is of level 1 $\Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1 \Rightarrow$ best guess is 1

$1 - f(1)$: agent 1 is of level $k \geq 2 \Rightarrow s_1^{(1)} = s_2^{(1)} = 0 \Rightarrow$ best guess is 0

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	Agent 1	Agent 2
Signals:	$\{0, 0\}$	$\{1\}$
Period 1:	0	1
Period 2:	0	1 $\leftarrow$ since $3p + q > 2$
Period 3:	0	1 $\leftarrow$ permanent disagreement if $f(1) > \frac{q-p}{2(p+q-1)}$
	$\vdots$	$\vdots$
	0	1

Assume $f(0) = 0$, $3p + q > 2$

Agent 2 knows:

- level 1 agent 1 stops updating after period 1
- level $k \geq 2$ agent 1 stops updating after period 2
- $\Rightarrow$ agent 1 stops updating after period 2

$\Rightarrow$ agent 2 stops updating after period 3

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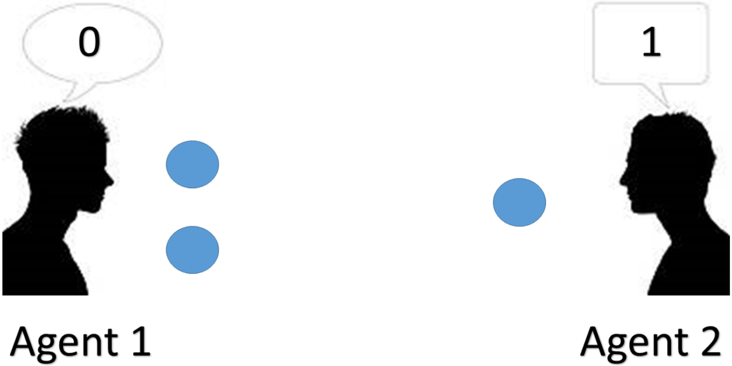

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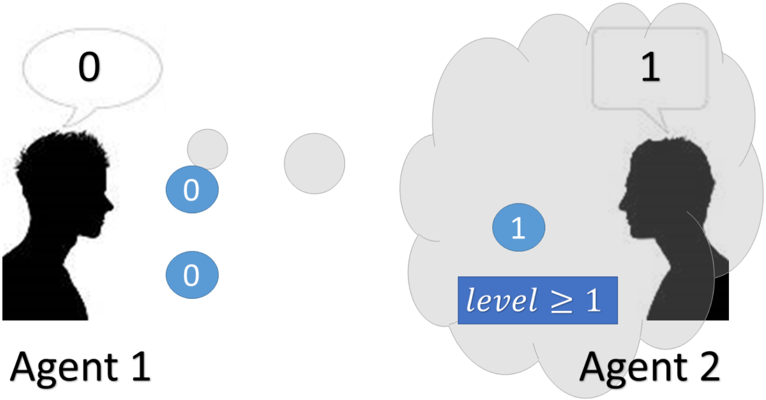

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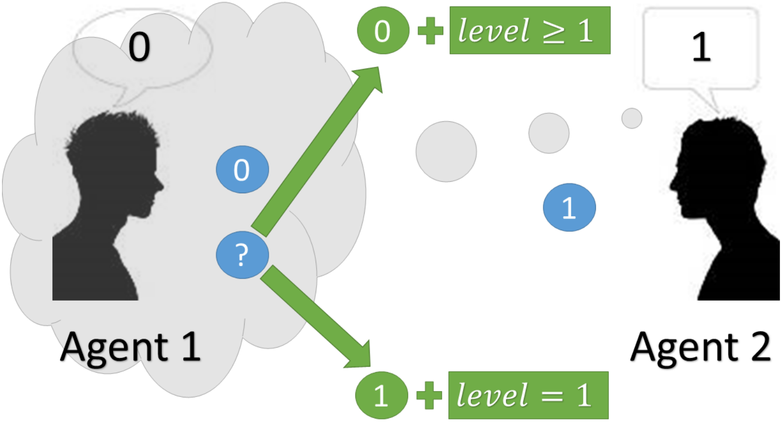

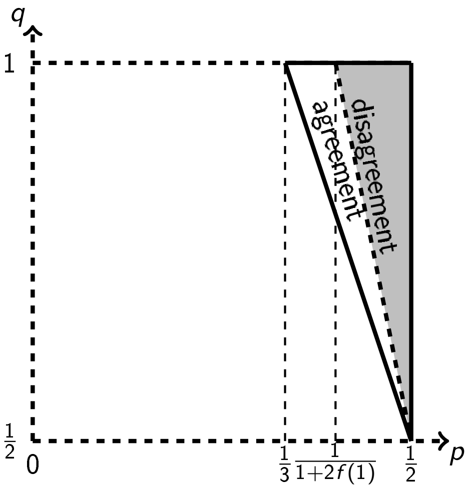

Recall:
$\mathbb{P}[\theta = 1] = p$
$\mathbb{P}[s = \theta \mid \theta] = q$

Disagreement happens

1. if the size of the doubt $f(1)$ is sufficiently high (at least $f(1) > 0.5$)
2. even if the cost of mistake is high: if $p \approx 0.5$, then

$$\text{cost} = \underbrace{q}_{\text{correct belief about } \theta = 0} - \underbrace{0.5}_{\text{belief of indifference}}$$

Example 2 raises three questions:

1. Do we need to have high doubt about the other agent's rationality to get disagreement?

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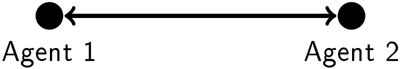
Example 2 raises three questions:

1. Do we need to have high doubt about the other agent's rationality to get disagreement? $\leftarrow$ NO (Example 3)
2. Can we get disagreement when $f(1) = 0$? $\leftarrow$ YES (Example 4)
3. Is it necessary for at least one agent to have more than one signal? $\leftarrow$ NO, but we need an incomplete network for it (Example 4)

Example 3



Example 3

Agent 1 — Agent 2

Signals: $\{0,0\}$ $\{1,1\}$

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Period 1:	0	1

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	$\vdots$	
	0	

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	Agent 1	Agent 2
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Period 1:	0	1
Period 2:	0	
Period 3:	0	
	$\vdots$	
	0	

Assume $f(0) = 0$

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	Agent 1	Agent 2
Signals:	$\{0,0\}$	$\{1,1\}$
Period 1:	0	1
Period 2:	0	1
Period 3:	0	
	$\vdots$	
	0	

Assume $f(0) = 0$

Agent 2: agent 1 is of level $k \geq 1 \Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1$

$$\mathbb{P}\left[\theta = 1 \mid s_1^{(2)} = s_2^{(2)} = 1,\ s_1^{(1)} + s_2^{(1)} \leq 1\right] > \frac{1}{2}$$

In contrast to Example 2, we don't need additional condition $3p + q > 2$ to guarantee 1

Example 3

	Agent 1	Agent 2
Signals:	$\{0,0\}$	$\{1,1\}$
Period 1:	0	1
Period 2:	0	1
Period 3:	0	1 $\longleftarrow$ if $f(1) > \frac{1}{2\left(\frac{p+q-1}{(1-2p)(1-q)q}+1\right)}$
	$\vdots$	
	0	

Assume $f(0) = 0$

$f(1)$ → agent 1 is of level 1 $\Rightarrow s_1^{(1)} + s_2^{(1)} \leq 1 \Rightarrow$ best guess is 1

$1 - f(1)$ → agent 1 is of level $k \geq 2 \Rightarrow s_1^{(1)} = s_2^{(1)} = 0 \Rightarrow$ best guess is 0

Example 3

	Agent 1	Agent 2
Signals:	$\{0, 0\}$	$\{1, 1\}$
Period 1:	0	1
Period 2:	0	1
Period 3:	0	1
	$\vdots$	$\vdots$
	0	1

permanent disagreement if $f(1) > \frac{1}{2\left(\frac{p+q-1}{(1-2p)(1-q)q}+1\right)}$

Assume $f(0) = 0$

Agent 2 knows:

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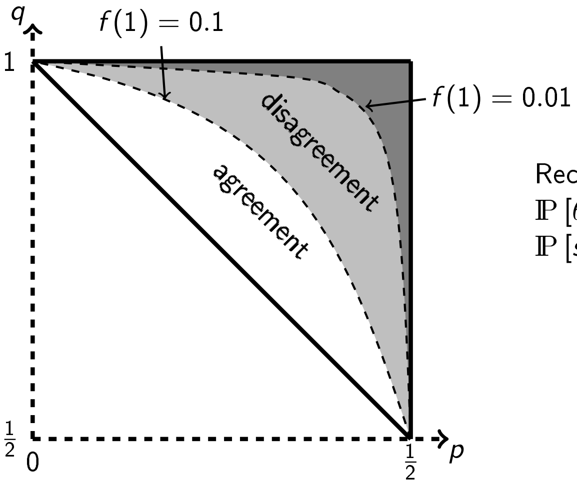
- level $k \geq 2$ agent 1 stops updating after period 2
- $\Rightarrow$ agent 1 stops updating after period 2

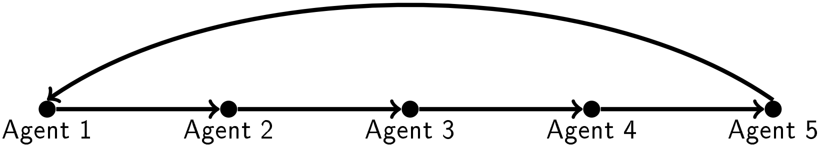
$\Rightarrow$ agent 2 stops updating after period 3



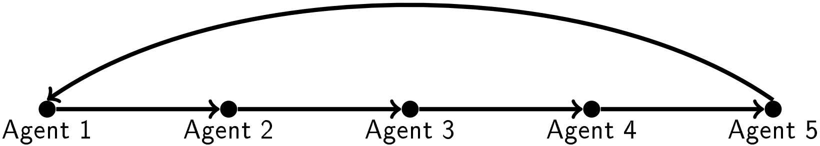
Recall:
$\mathbb{P}[\theta = 1] = p$
$\mathbb{P}[s = \theta \mid \theta] = q$

The size of the doubt $f(1)$ can be arbitrary small: for any $f(1) \in (0, 1)$ we can find $p \in (0, 0.5)$ and $q \in (0.5, 1)$ such that $p + q > 1$ and the agents permanently disagree with each other for some signals realizations.

Example 4

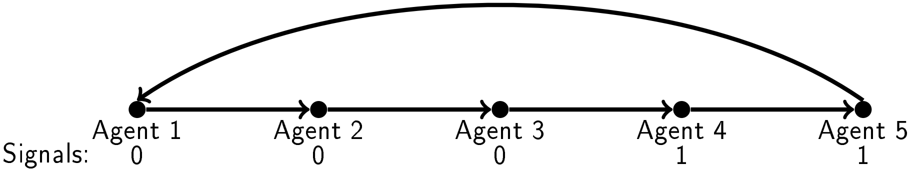


Example 4



Assume $f(0) = f(1) = f(2) = 0$

Example 4



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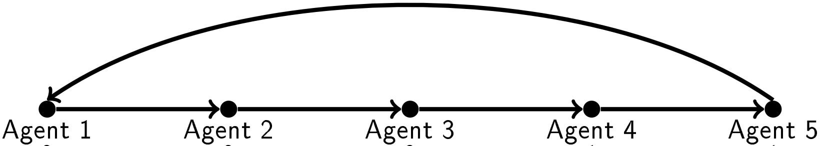
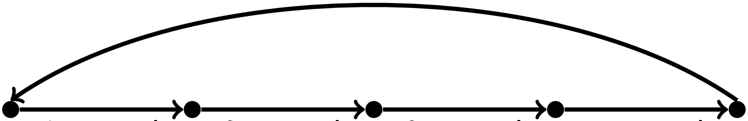


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1

Assume $f(0) = f(1) = f(2) = 0$

Example 4

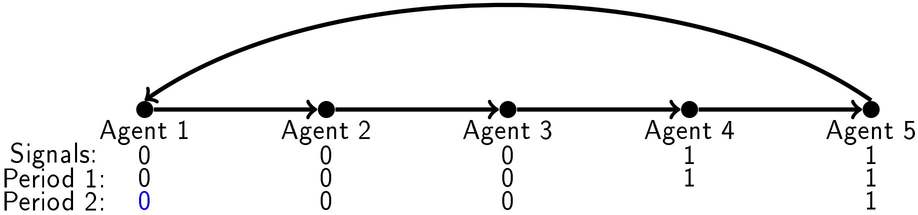


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:		0	0		1

Assume $f(0) = f(1) = f(2) = 0$

Agents 2, 3 and 5 infer that their neighbors got exactly the same signal as they did. $\Rightarrow$ Their guesses do not change.

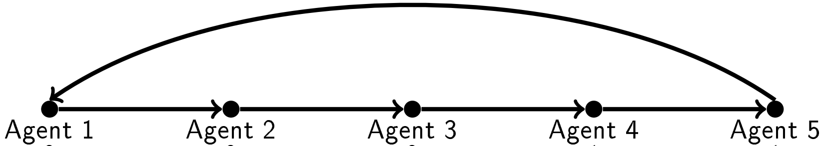
Example 4



Assume $f(0) = f(1) = f(2) = 0$

Agent 1 reasons that agent 5 got signal 1, but since the prior is biased towards 0, agent 1 still reports 0 in period 2.

Example 4

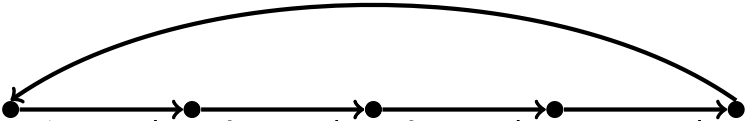


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1

Assume $f(0) = f(1) = f(2) = 0$

Agent 4 is in the reversed situation and therefore he is the only one who changes his guess.

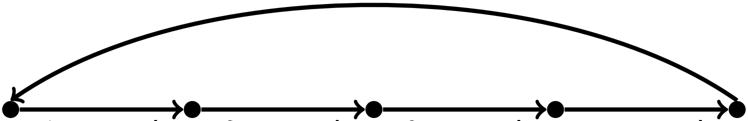
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:					
Period 4:					

Assume $f(0) = f(1) = f(2) = 0 \Rightarrow$ It is common knowledge that everybody is behaving rationally in periods 1,2 and 3.

Example 4

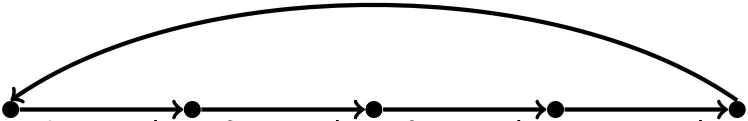


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:		0	0	0	
Period 4:					

Assume $f(0) = f(1) = f(2) = 0$

In period 3, agents 2, 3 and 4 observed only 0-s from their neighbors and therefore have no reason to change their guesses to 1.

Example 4



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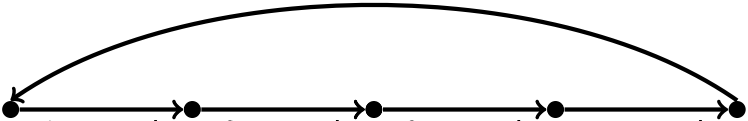
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	
Period 4:					

Assume $f(0) = f(1) = f(2) = 0$

Agent 1 infers that both agents 4 and 5 got signals 1 (otherwise agent 5 would have switched to 0 in period 2). $\Rightarrow$ Agent 1 reports 1 in period 3.

Example 4

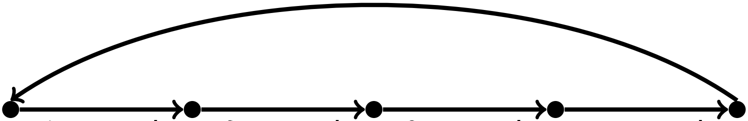


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:					

Assume $f(0) = f(1) = f(2) = 0$

Agent 5 infers that agent 4 got 1 but agent 3 got 0. However, since agent 5 himself got 1, he would not switch to 0 in period 3.

Example 4

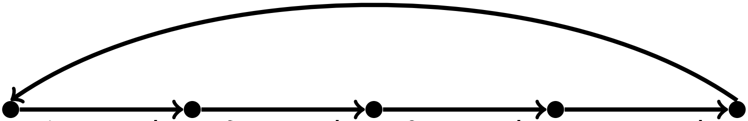


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1		0	0	

Assume $f(0) = f(1) = f(2) = 0$

In period 4, agents 3 and 4 guess 0 as they received 0 from their neighbors. Similarly, agent 1 does not change his guess.

Example 4

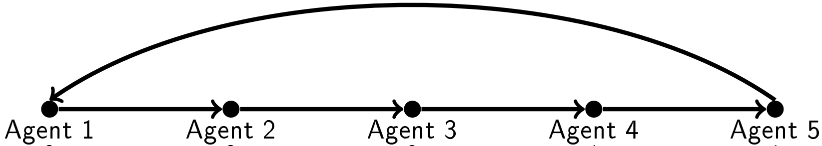


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	

Assume $f(0) = f(1) = f(2) = 0$

Agent 2 infers that agent 5 must have guessed 1 in periods 1 and 2 because that is the only situation when a rational agent would switch to 1 in period 3. That implies both agents 4 and 5 got 1. So, agent 2 now knows four signals: 0 (his own), 0 (received by agent 1), 1 (received by agent 5) and 1 (received by agent 4). Thus, he guesses 0 in period 4.

Example 4

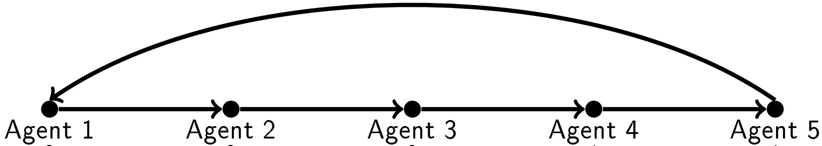


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1

Assume $f(0) = f(1) = f(2) = 0$

Agent 5 infers no new information from his neighbor' guess in period 2 since combination 1-0-1 simply impossible to hear from a rational agent. Thus, agent 5 guesses 1 in period 4.

Example 4

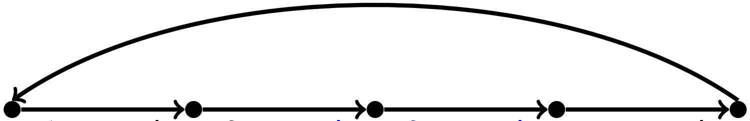


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
	⋮	⋮	⋮	⋮	⋮

Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

Example 4



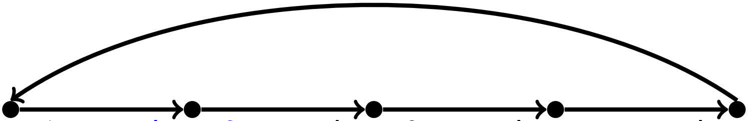
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
	⋮	⋮	⋮	⋮	⋮

Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

- Agents 1, 3 and 4 have no reason to change their guesses since they agree with their neighbors.

Example 4



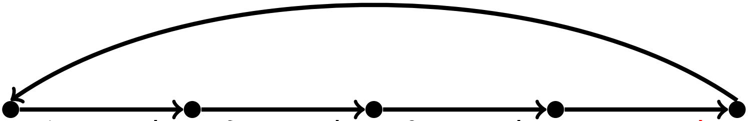
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
	⋮	⋮	⋮	⋮	⋮

Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

- Agents 1, 3 and 4 have no reason to change their guesses since they agree with their neighbors.
- Agent 2 is correct, so his disagreement is “justified”.

Example 4



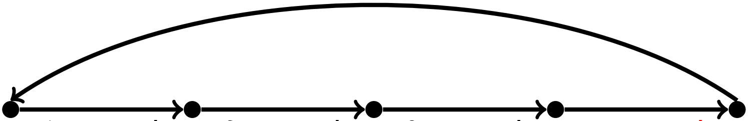
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
	⋮	⋮	⋮	⋮	⋮

Assume $f(0) = f(1) = f(2) = 0$

Claim: Under some assumptions, no agent will change their guess starting from period 4.

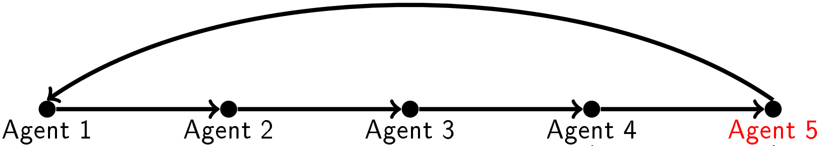
- Agents 1, 3 and 4 have no reason to change their guesses since they agree with their neighbors.
- Agent 2 is correct, so his disagreement is “justified”.
- What is surprising in this example is the persistent disagreement of agent 5. (skip proof middle step)

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:					1
Period 1:				1	1
Period 2:				0	1
Period 3:				0	1
Period 4:				0	1
Period 5:					

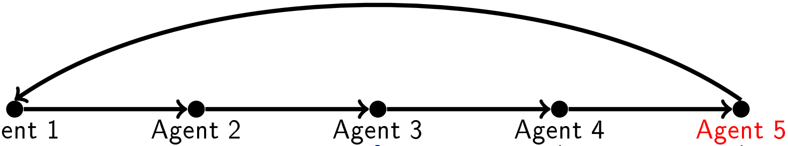
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:				1	1
Period 1:				1	1
Period 2:				0	1
Period 3:				0	1
Period 4:				0	1
Period 5:					

Agent 5 knows that agent 4 reports his signal in period 1. $\Rightarrow$ Agent 4's signal is 1.

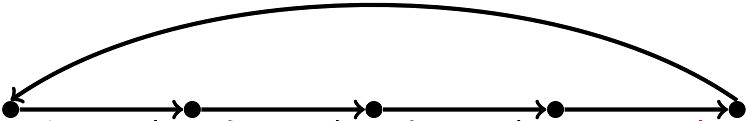
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:				0	1
Period 3:				0	1
Period 4:				0	1
Period 5:					

Agent 4's report 0 in period 2 indicates that agent 3 reported 0 in period 1 and therefore has signal 0.

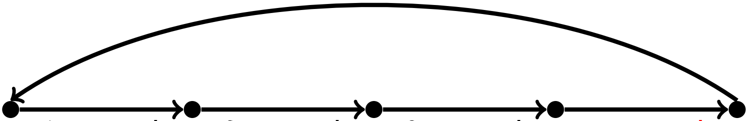
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:			0	0	1
Period 3:				0	1
Period 4:				0	1
Period 5:					

No agent of level $k \geq 3$ would ever report 1 in period 2 if he reported 0 in period 1. $\Rightarrow$ Agent 3 reported 0 in period 2 as well.

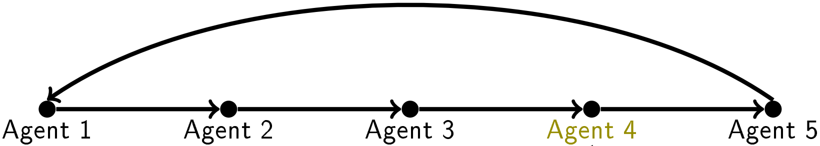
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:			0	0	1
Period 3:			1?	0	1
Period 4:				0	1
Period 5:					

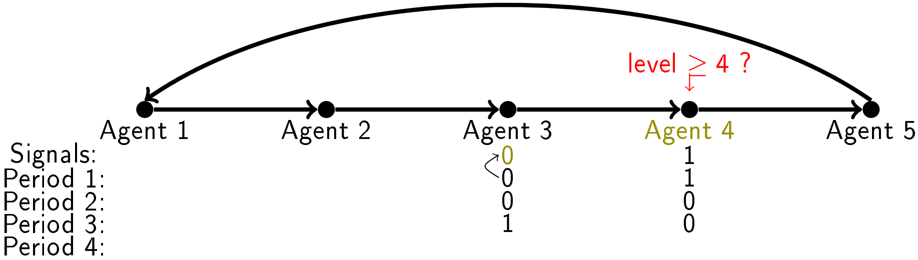
Suppose agent 3 reports 1 in period 3.

Example 4



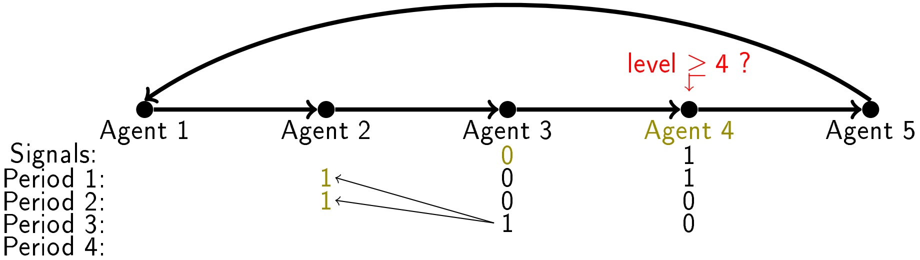
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:				1	
Period 1:			0	1	
Period 2:			0	0	
Period 3:			1	0	
Period 4:					

Example 4



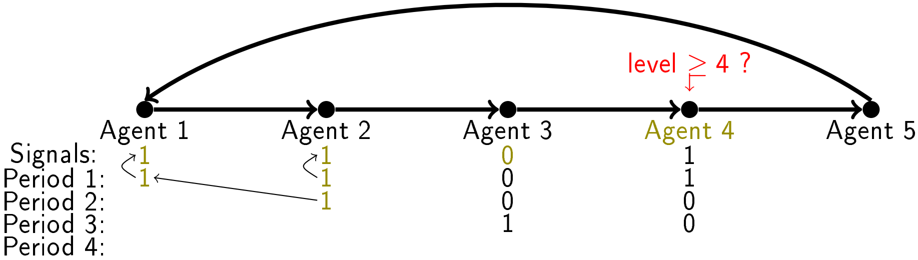
If agent 4 is of level 4 or higher, he is rational in period 4.

Example 4



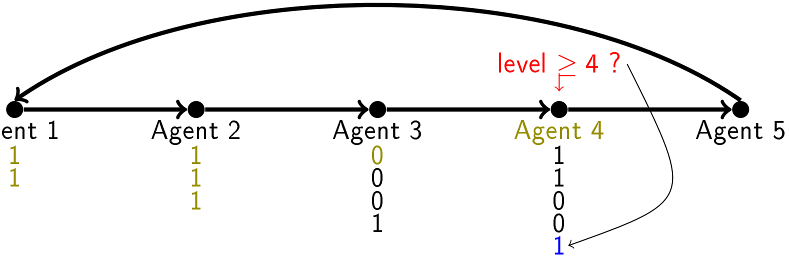
If agent 4 is of level 4 or higher, he is rational in period 4.

Example 4



If agent 4 is of level 4 or higher, he is rational in period 4.

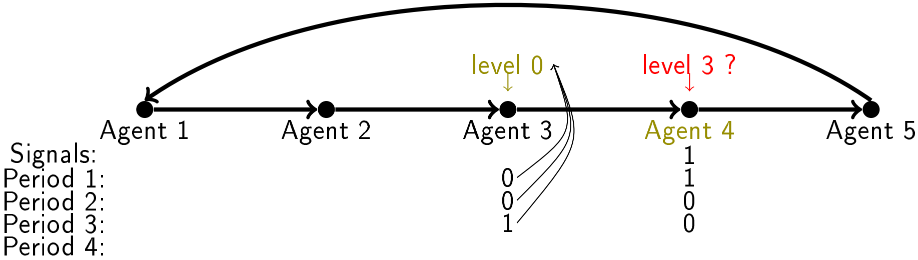
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	1	1	0	1	
Period 1:	1	1	0	1	
Period 2:		1	0	0	
Period 3:			1	0	
Period 4:				1	

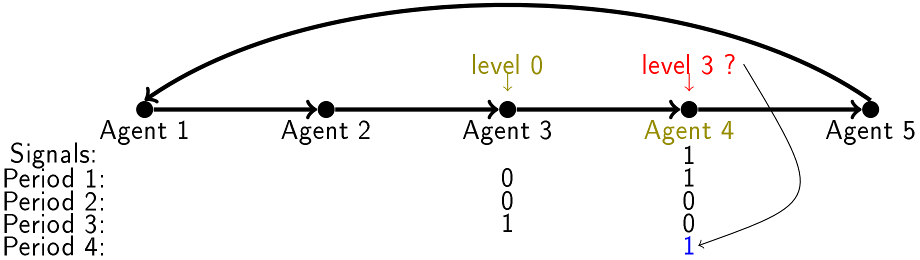
Agent 4 knows three signals: 0 (his own), 1 (agent 2), 1 (agent 1)
$\Rightarrow$ his best guess is 1

Example 4



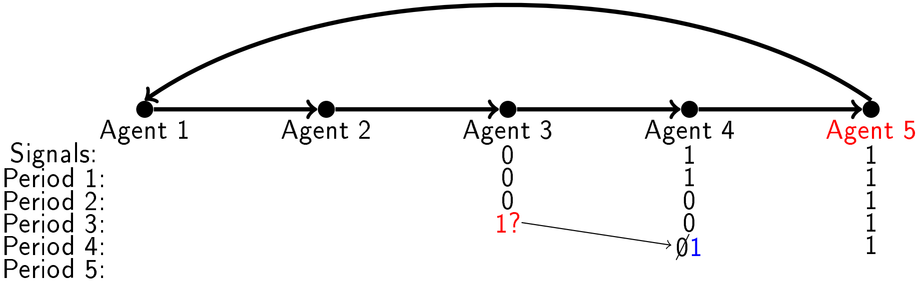
If agent 4 is of level 3, he believes that agent 3 is of level 0: behavior 0-0-1 is inconsistent with neither level 2 nor with level 1

Example 4



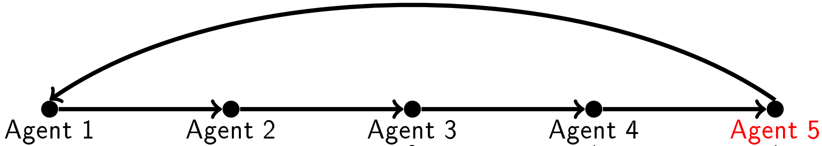
Agent 3 is of level 0, agent 4's own signal is 1 ⇒ best guess is 1

Example 4



Suppose agent 3 reports 1 in period 3. ⇒ Agent 3 reports 1 in period 4. ⇒ Contradiction.

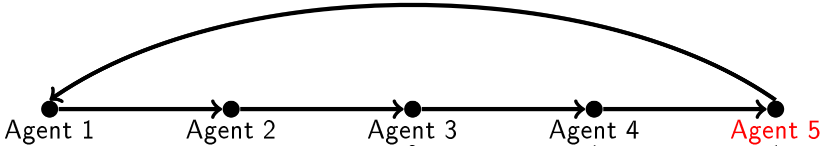
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:			0	0	1
Period 3:			0 ←	0	1
Period 4:				0	1
Period 5:					

Suppose agent 3 reports 1 in period 3. ⇒ Agent 3 reports 1 in period 4. ⇒ Contradiction.

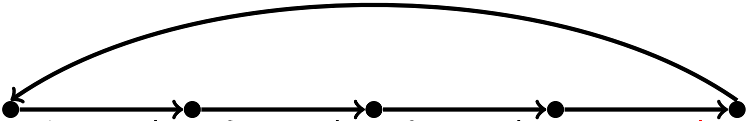
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:				0	1
Period 5:					

Agent 3 is of level $\geq 3 \Rightarrow$ Agent 2 reports 0 in period 2

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:				0	1
Period 5:					1

since $3p + q > 2$

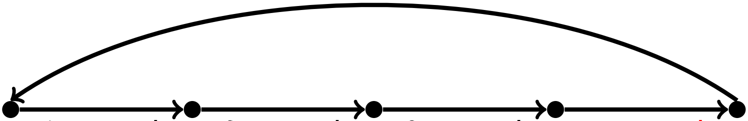
Assume $3p + q > 2$

That is all information agent 5 has by period 5. He knows that agent 3 has signal 0, agent 4 has signal 1 and agents 1 and 2 together have at least one 0 signal. So, agent 5's posterior belief is

$$\mathbb{P}\left[\theta = 1 \mid s^{(1)} + s^{(2)} \leq 1,\ s^{(3)} = 0,\ s^{(4)} = s^{(5)} = 1\right] > \frac{1}{2}$$

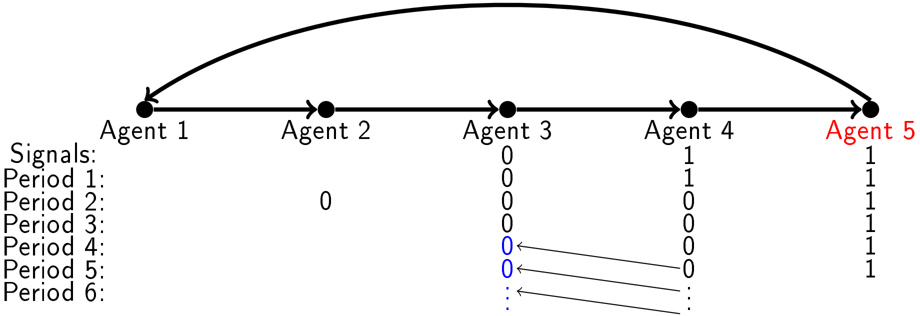
$$\Leftrightarrow \quad 3p + q > 2$$

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:				0	1
Period 5:				0	1
Period 6:				⋮	

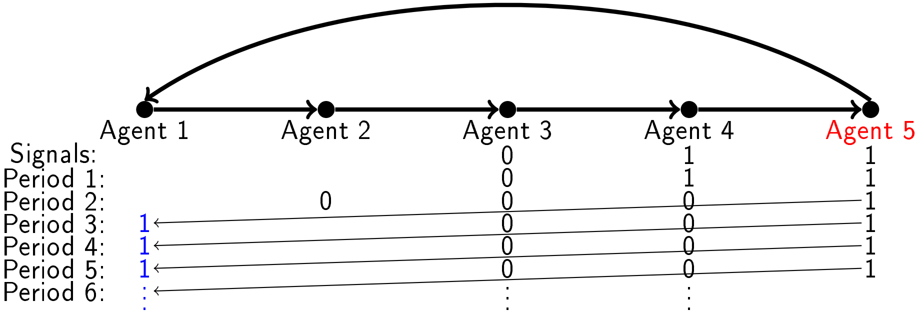
Example 4



Agent 3 must have reported all zeros in all previous periods:
agent 3 reports 1 in period $t-2 \Rightarrow$ in period $t-1$ agent 4 reasons

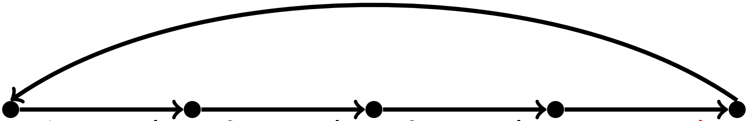
- either agent 3 is “smart” $\Rightarrow$ agent 4 should “believe” him and report 1
- or agent 3 is “stupid” $\Rightarrow$ agent 4 should “ignore” him and report 1 (his own signal)

Example 4



Agent 5 also knows that as long as he keeps reporting 1, agent 1 will guess 1 starting from period 3.

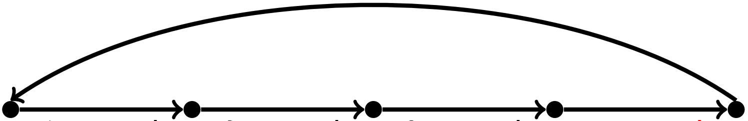
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:		1?	0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:	1		0	0	1
Period 4:	1		0	0	1
Period 5:	1		0	0	1
Period 6:	⋮		⋮	⋮	

Suppose agent 2 received signal 1.

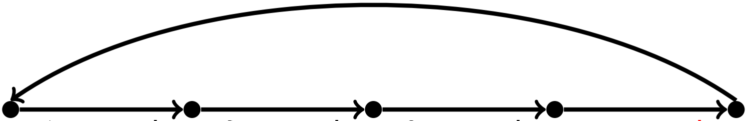
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:		1?	0	1	1
Period 1:		1	0	1	1
Period 2:		0	0	0	1
Period 3:	1		0	0	1
Period 4:	1		0	0	1
Period 5:	1		0	0	1
Period 6:	⋮		⋮	⋮	

Suppose agent 2 received signal 1.

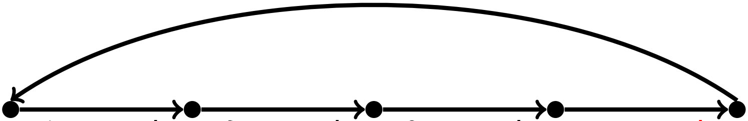
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:		1?	0	1	1
Period 1:	0	1	0	1	1
Period 2:		0	0	0	1
Period 3:	1		0	0	1
Period 4:	1		0	0	1
Period 5:	1		0	0	1
Period 6:	⋮		⋮	⋮	

Suppose agent 2 received signal 1.

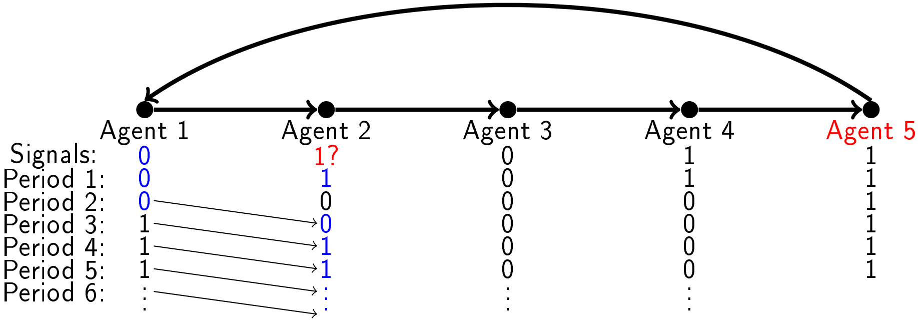
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	1?	0	1	1
Period 1:	0	1	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1		0	0	1
Period 4:	1		0	0	1
Period 5:	1		0	0	1
Period 6:	⋮		⋮	⋮	

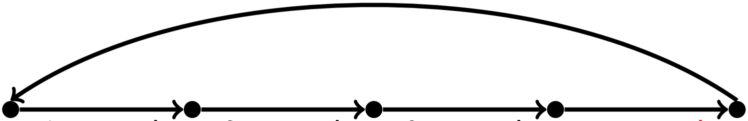
Suppose agent 2 received signal 1.

Example 4



Suppose agent 2 received signal 1.
Similar to agent 4, agent 2 repeats the guess of his neighbor starting from period 3.

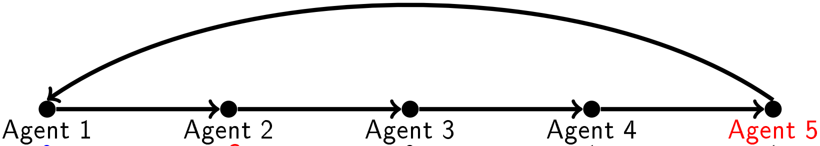
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	1?	0	1	1
Period 1:	0	1	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	1	0	0	1
Period 5:	1	1	0	0	1
Period 6:	⋮	⋮	⋮	⋮	

$$(\underbrace{s^{(1)}, s^{(2)}, s^{(3)}, s^{(4)}, s^{(5)},}_{\text{signals}} \underbrace{k_1, k_2, k_3, k_4}_{\text{levels of reasoning}}) \quad \Rightarrow \quad \text{all guesses}$$

Example 4



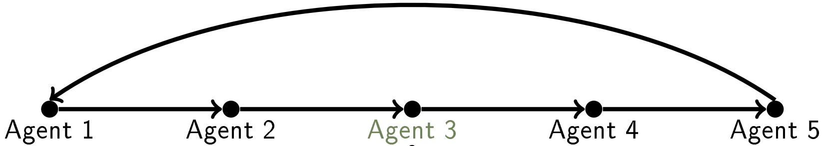
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	1?	0	1	1
Period 1:	0	1	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	1	0	0	1
Period 5:	1	1	0	0	1
Period 6:	⋮	⋮	⋮	⋮	

$$(\underbrace{s^{(1)}, s^{(2)}, s^{(3)}, s^{(4)}, s^{(5)},}_{\text{signals}} \underbrace{k_1, k_2, k_3, k_4}_{\text{levels of reasoning}}) \quad \Rightarrow \quad \text{all guesses}$$

Suppose this configuration is correct.

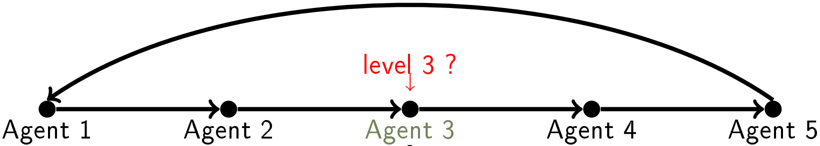
- signals are fixed
- agent 5 cannot exclude any level $k_i \geq 3$ of reasoning for agents 1, 2 and 4
- what about agent 3?

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0		
Period 1:		1			
Period 2:		0			
Period 3:		0			
Period 4:		1			
Period 5:		1			
Period 6:		⋮			

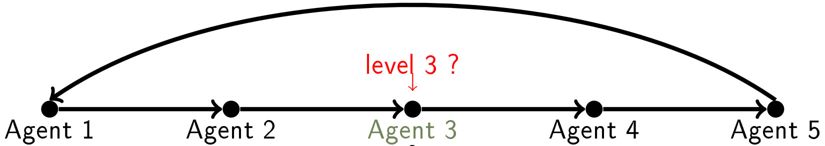
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0		
Period 1:		1			
Period 2:		0			
Period 3:		0			
Period 4:		1			
Period 5:		1			
Period 6:		⋮			

Suppose agent 3 is of level 3.

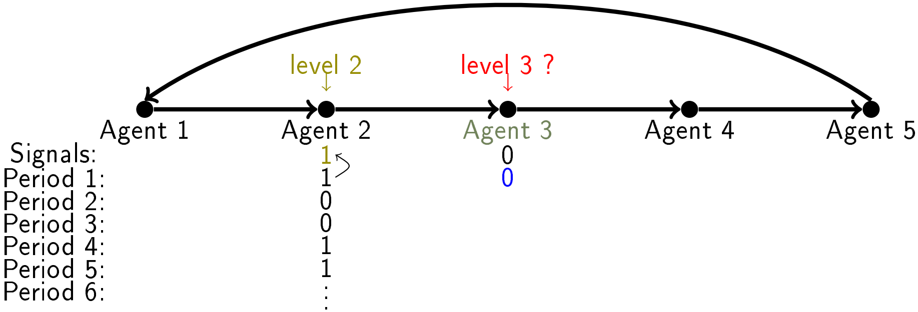
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0		
Period 1:		1	0		
Period 2:		0			
Period 3:		0			
Period 4:		1			
Period 5:		1			
Period 6:		⋮			

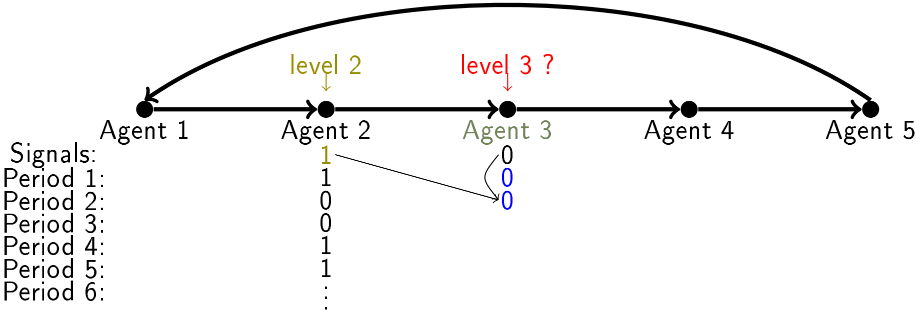
In period 1, agent 3 reports 0, his signal.

Example 4



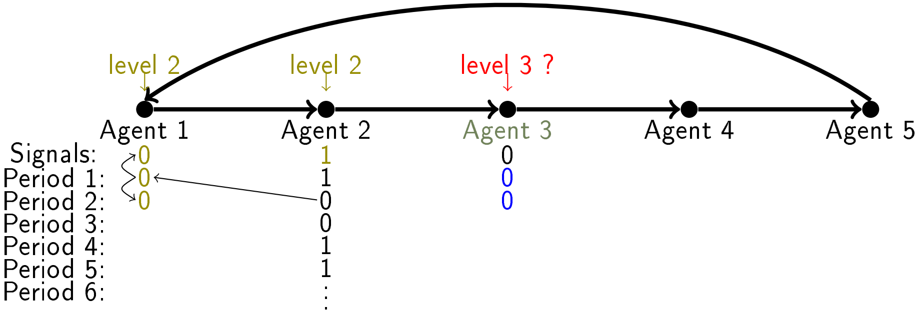
In period 2, thinking that agent 2 is of level 2, agent 3 infers that agent 2 has 1

Example 4



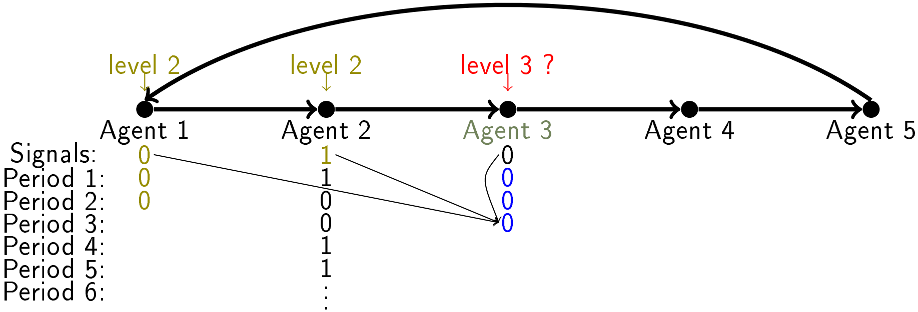
In period 2, thinking that agent 2 is of level 2, agent 3 infers that agent 2 has 1 but it is still optimal to report 0 since the prior is biased towards 0.

Example 4



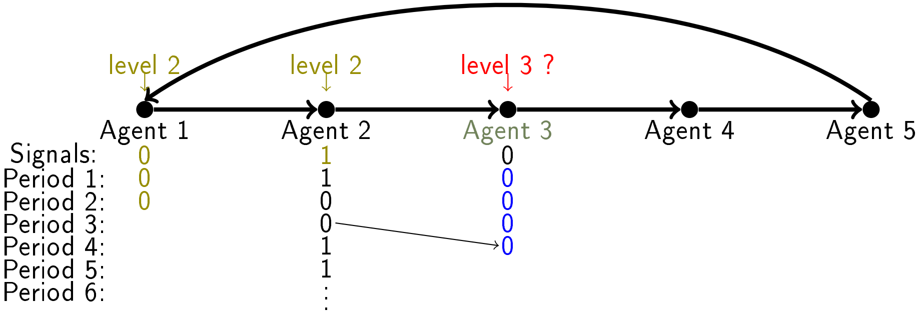
In period 3, agent 3 infers that agent 1 has signal 0

Example 4



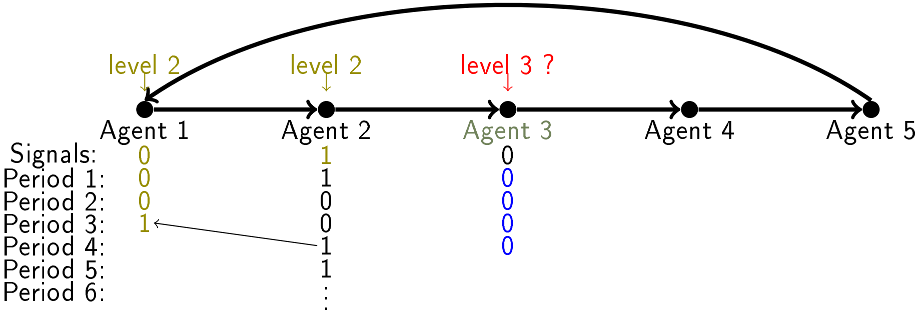
In period 3, agent 3 infers that agent 1 has signal 0, so he reports 0 again.

Example 4



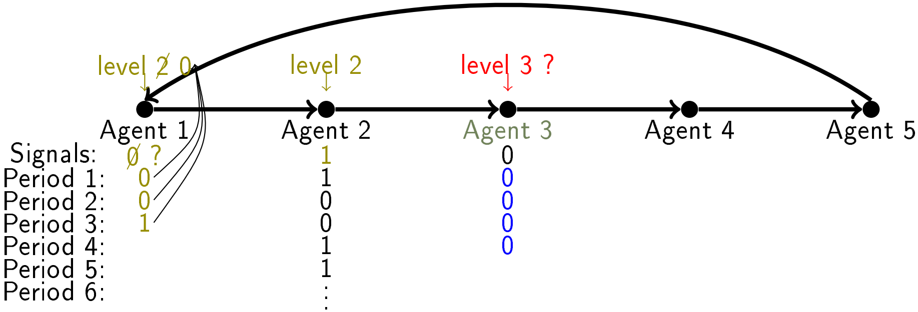
In period 4, agent 4 gets no reason to change the guess.

Example 4



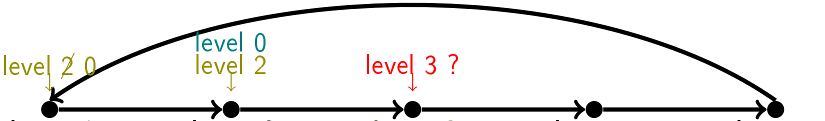
If agent 2 is of level 2, then agent 1 reported 1 in period 3

Example 4



If agent 2 is of level 2, then agent 1 reported 1 in period 3 ⇒ Agent 1 must be of level 0

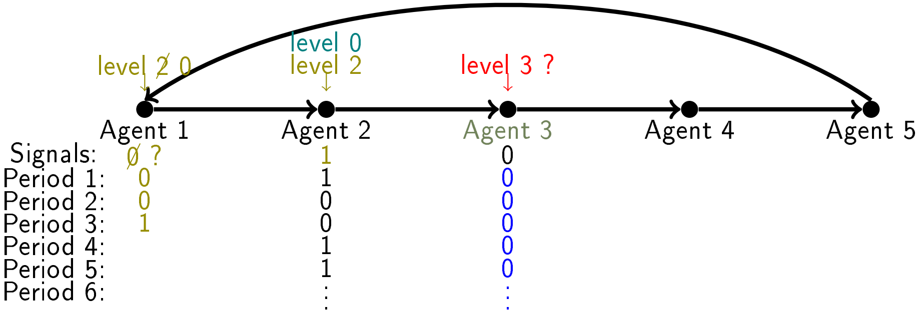
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	~~0~~ ?	1	0		
Period 1:	0	1	0		
Period 2:	0	0	0		
Period 3:	1	0	0		
Period 4:		1	0		
Period 5:		1			
Period 6:		⋮			

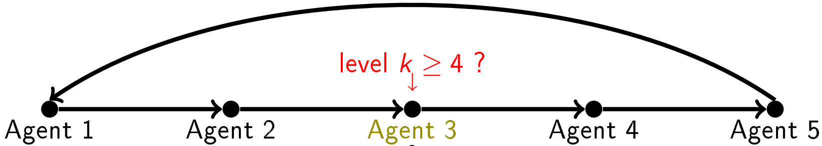
If agent 2 is of level 2, then agent 1 reported 1 in period 3 ⇒
Agent 1 must be of level 0 or agent 2 is of level 0.

Example 4



If agent 2 is of level 2, then agent 1 reported 1 in period 3 ⇒ Agent 1 must be of level 0 or agent 2 is of level 0. Either way, the best guess is 0.

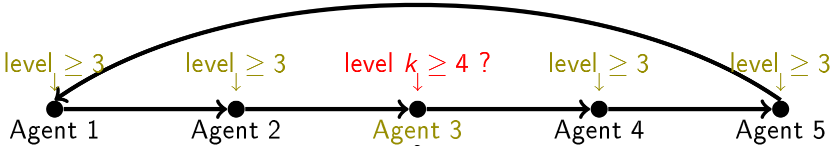
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0		
Period 1:		1			
Period 2:		0			
Period 3:		0			
Period 4:		1			
Period 5:		1			
Period 6:		⋮			

Suppose agent 3 is of level $k \geq 4$.

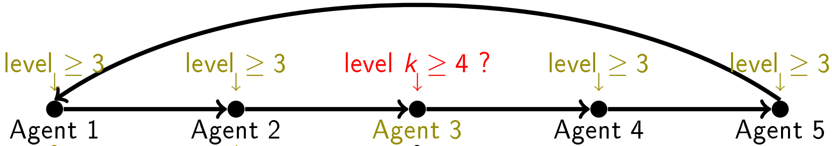
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0		
Period 1:		1			
Period 2:		0			
Period 3:		0			
Period 4:		1			
Period 5:		1			
Period 6:		⋮			

Agent 3 thinks that everybody else is of level at least 3.

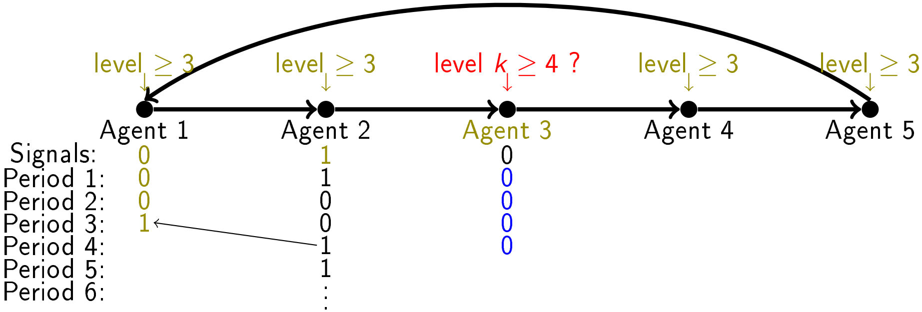
Example 4



	Agent 1	Agent 2	Agent 3
Signals:	0	1	0
Period 1:	0	1	0
Period 2:	0	0	0
Period 3:		0	0
Period 4:		1	0
Period 5:		1	
Period 6:		⋮	

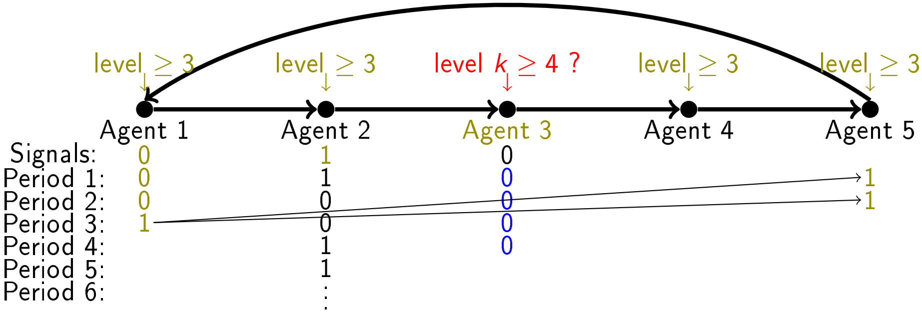
Up to period 4, level $k \geq 4$ agent 3 reasons exactly the same as level 3 agent 3.

Example 4



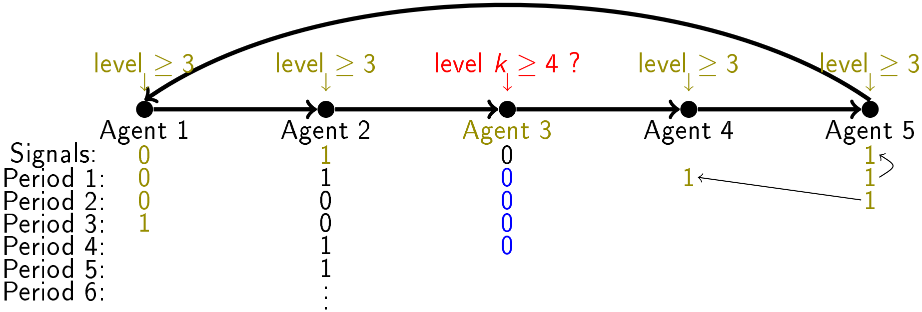
In period 5, agent 3 concludes that agent 1 reported 1 in period 3 (since agent 2 is of level 3 or higher)

Example 4



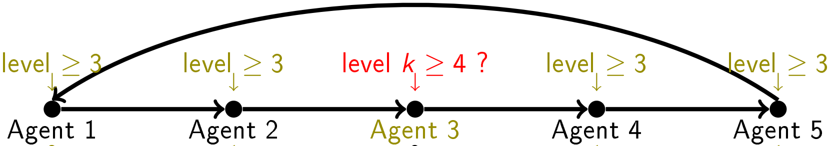
In contrast to level 3 agent 3, level $k \geq 4$ agent 3 makes the right conclusion in period 5: agent 1 reports 0-0-1 because agent 5 reports 1-1.

Example 4



Being of level at least 3, agent 5 reports 1-1 if and only if he gets signal 1 and observes 1 from agent 4.

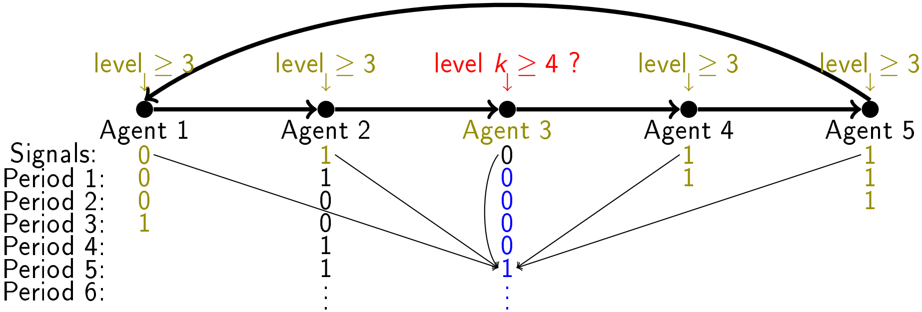
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	1	0	1	1
Period 1:	0	1	0	1	1
Period 2:	0	0	0		1
Period 3:	1	0	0		
Period 4:		1	0		
Period 5:		1			
Period 6:		⋮			

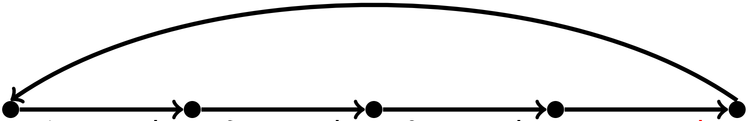
Since agent 4 follows his own signal in period 1, agent 3 concludes that agent 4 has signal 1.

Example 4



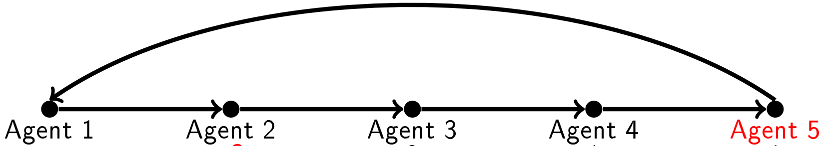
Thus, starting from period 5, agent 3 knows that among five signal three are 1, so the best guess is 1. Since agent 2 continues to report 1, there is no reason for agent 3 to change his mind.

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:	1		0	0	1
Period 4:	1		0	0	1
Period 5:	1		0	0	1
Period 6:	⋮		⋮	⋮	

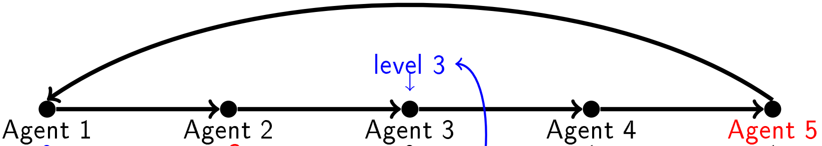
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:		1?	0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:	1		0	0	1
Period 4:	1		0	0	1
Period 5:	1		0	0	1
Period 6:	⋮		⋮	⋮	

Assume agent 2 got signal 1.

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	1?	0	1	1
Period 1:	0	1	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	1	0	0	1
Period 5:	1	1	0	0	1
Period 6:	⋮	⋮	⋮	⋮	

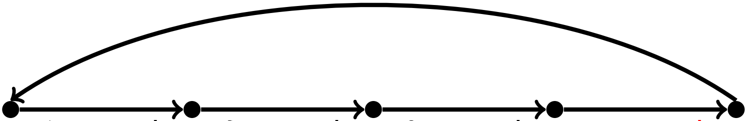
Assume agent 2 got signal 1.

<u>Period 6</u>: Agent 5 concludes that agent 3 reported all 0s in periods 1-4
$\Rightarrow$ Consistent with any level $k \geq 3$ of reasoning for agent 3

<u>Period 7</u>: Agent 5 concludes that agent 3 reported all 0s in periods 1-5
$\Rightarrow$ Consistent only with level 3 behavior for agent 3

▸ to the end

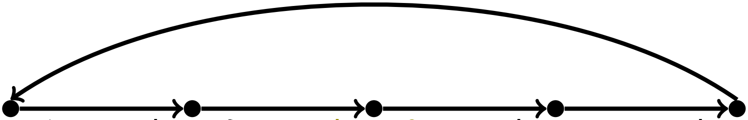
Example 4



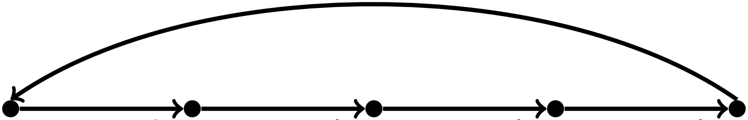
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	1?	0	0	1	1
Period 1:	1	0	0	1	1
Period 2:	1	0	0	0	1
Period 3:	1	1	0	0	1
Period 4:	1	1	0	0	1
Period 5:	1	1	0	0	1
Period 6:	⋮	⋮	⋮	⋮	

Assume agent 1 got signal 1.

Configuration is consistent with any level $k \geq 3$ of reasoning for agents 1, 2 and 4.
What about agent 3?

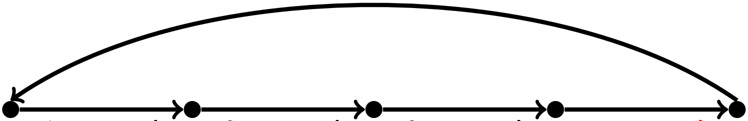


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	1	0	0	1	1
Period 1:	1	0	0	1	1
Period 2:	1	0	0	0	1
Period 3:	1	1	0	0	1
Period 4:	1	1	0	0	1
Period 5:	1	1	0	0	1
Period 6:	⋮	⋮	⋮	⋮	



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
Period 5:	1	0	0	0	1
Period 6:	⋮	⋮	⋮	⋮	⋮

Example 4

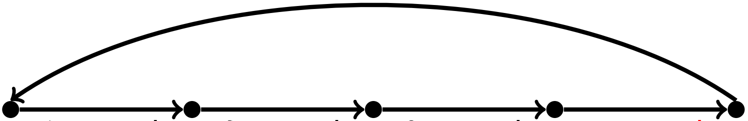


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	1?	0	0	1	1
Period 1:	1	0	0	1	1
Period 2:	1	0	0	0	1
Period 3:	1	1	0	0	1
Period 4:	1	1	0	0	1
Period 5:	1	1	0	0	1
Period 6:	⋮	⋮	⋮	⋮	

Assume agent 1 got signal 1.

$\Rightarrow$ Consistent with any level $k \geq 3$ of reasoning for agents 1, 2, 3 and 4

Example 4



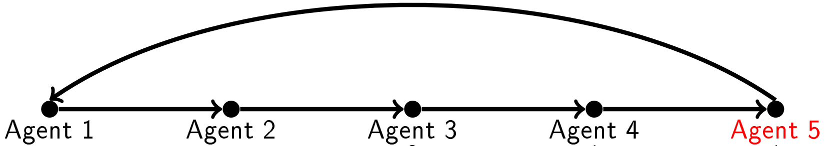
	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0?	0?	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
Period 5:	1	0	0	0	1
Period 6:	⋮	⋮	⋮	⋮	

Assume both agents 1 and 2 received signals 0.

This is in fact the situation that is realized (though agent 5 does not know it).

⇒ Consistent with any level $k \geq 3$ of reasoning for agents 1, 2, 3 and 4

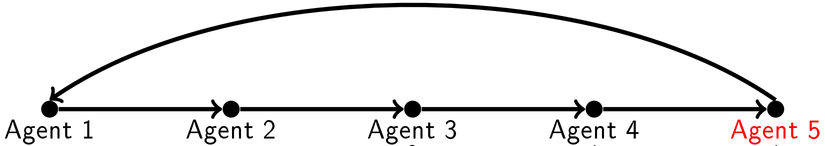
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:			0	0	1
Period 5:				0	1
Period 6:					1

In period 6, agent 5 does not receive any new information (he anticipates agent 4 to report 4) $\Rightarrow$ agent 5 does not change his guess

Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:			0	0	1
Period 5:			0	0	1
Period 6:				0	1
Period 7:					

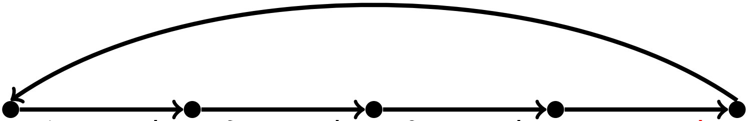
In period 7, agent 5 excludes configuration

$$s^{(1)} = 0,\ s^{(2)} = 1,\ s^{(3)} = 0,\ s^{(4)} = 1,\ s^{(5)} = 1,\ k_3 \geq 4$$

$f(3)$ → agent 3 is of level 3 $\Rightarrow s^{(1)} + s^{(1)} \leq 1 \Rightarrow$ best guess is 1

$1 - f(3)$ → agent 3 is of level $k \geq 4 \Rightarrow s_2^{(1)} = 0 \Rightarrow$ best guess is 0

Example 4

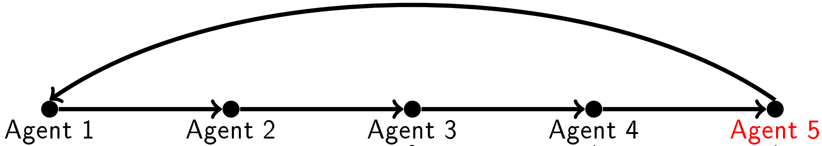


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:			0	0	1
Period 5:			0	0	1
Period 6:				0	1
Period 7:					1

In period 7, agent 5 guesses 1 if

$$f(3) > \frac{1-2p}{p+q-1}$$

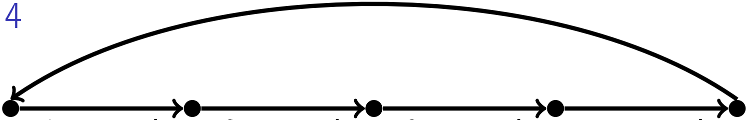
Example 4



	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:			0	1	1
Period 1:			0	1	1
Period 2:		0	0	0	1
Period 3:			0	0	1
Period 4:			0	0	1
Period 5:			0	0	1
Period 6:				0	1
Period 7:					1
					⋮
					1

Starting from period 8, no new information received (agent 5 anticipates receiving 0 from his neighbor) $\Rightarrow$ disagreement is permanent

Example 4

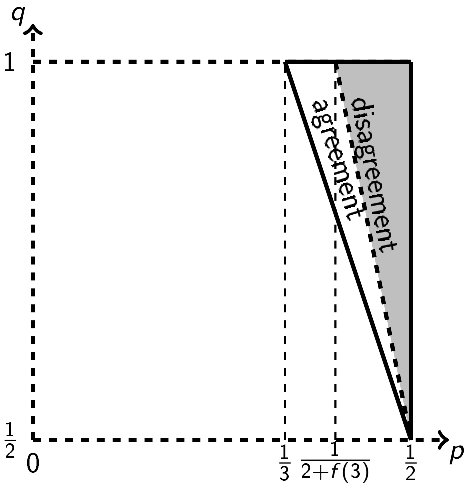


	Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
Signals:	0	0	0	1	1
Period 1:	0	0	0	1	1
Period 2:	0	0	0	0	1
Period 3:	1	0	0	0	1
Period 4:	1	0	0	0	1
Period 5:	1	0	0	0	1
Period 6:	1	0	0	0	1
Period 7:	1	0	0	0	1
	⋮	⋮	⋮	⋮	⋮
	1	0	0	0	1

No matter how much time has passed, agent 5 cannot completely exclude the situation when agent 3 is of level 3 and therefore cannot exclude the situation when agent 2 has signal 1. ⇒ Agent 5 does not know that agent 2 disagrees with him.

Agent 2 does know that agent 5 disagrees with him but he does not know why this happens: it's because agent 5 thinks that agent 3 might be of level 3, or it's because agent 3 has signal 1.

Example 4



Recall:
$\mathbb{P}[\theta = 1] = p$
$\mathbb{P}[s = \theta \mid \theta] = q$

We assumed $f(0) = f(1) = f(2) = 0$
Disagreement can happen even for the smallest doubt $f(3)$!

Conclusion

Today: Examples

- Cognitive hierarchical model in networks
- Small doubt $\overset{\text{sometimes}}{\Rightarrow}$ permanent disagreement ($\Rightarrow$ trade)
- Idea: Agent who disagrees can never learn which of several scenarios is realized (learning stops in finite time).

Future research: General results